On Two-Directional Orthogonal Ray Graphs

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1. Introduction

A bipartite graph $G$ with a bipartition $(U, V)$ is called an orthogonal ray graph if there exist a family of non-intersecting rays (half-lines) $R_u, u \in U$, parallel to the $x$-axis in the $xy$-plane, and a family of non-intersecting rays $R_v, v \in V$, parallel to the $y$-axis such that for any $u \in U$ and $v \in V$, $(u, v) \in E(G)$ if and only if $R_u$ and $R_v$ intersect. An orthogonal ray graph $G$ is called a 2-directional orthogonal ray graph if $R_u = \{(x, b_u) \mid x \geq a_u\}$ for each $u \in U$, and $R_v = \{(a_v, y) \mid y \geq b_v\}$ for each $v \in V$, where $a_u$ and $b_v$ are real numbers for any $w \in U \cup V$. We introduced orthogonal ray graphs[8] in connection with defect tolerance schemes for nano-programmable logic arrays[7], [9]. In this paper, we provide four characterizations of 2-directional orthogonal ray graphs and their implications on the recognition and isomorphism problems of such graphs and an open question posed by Klinz, Rudolf, and Woeginger[3].

2. Preliminaries

Let $G$ be a bipartite graph with a bipartition $(U, V)$. A $(0, 1)$-matrix $M = [m_{ij}]$ is called a bipartite adjacency matrix of $G$ if the rows of $M$ correspond to the vertices of $U$, the columns of $M$ correspond to the vertices of $V$, and $m_{ij} = 1$ if and only if $(u_i, v_j) \in E(G)$, where $u_i \in U$ is a vertex corresponding to row $i$, and $v_j \in V$ is a vertex corresponding to column $j$.

Let $A$ be a matrix and $S$ be a set of matrices. $A$ is said to be $S$-freeable if there exist a permutation of the rows of $A$ and a permutation of the columns of $A$ such that the permuted matrix does not contain any $B \in S$ as a submatrix.

A bipartite graph $G$ with bipartition $(U, V)$ is said to be weakly orderable if there exist an ordering $(v_1, v_2, \ldots, v_{|V|})$ of $V$ and an ordering $(u_1, u_2, \ldots, u_{|U|})$ of $U$ such that for every $i, i', j, j'$ ( $1 \leq i < i' \leq |U|$, $1 \leq j < j' \leq |V|$), $(u_i, v_j) \in E(G)$ and $(u_i, v_{j'}) \in E(G)$ imply $(u_{i'}, v_j) \in E(G)$.

A graph $G$ is called a circular arc graph if there exists a collection of circular arcs $A_u, u \in V(G)$, on a fixed circle, such that two arcs $A_v$ and $A_w$ intersect if and only if $(v, w) \in E(G)$

A bipartite graph is chordal bipartite if it contains no cycle of length at least 6 as an induced subgraph.

An edge-asteroid is a set of edges $e_0, e_1, \ldots, e_{2k}$ such that for each $i = 0, 1, \ldots, 2k$, there is a path joining $e_i$ and $e_{i+1}$, and containing both $e_i$ and $e_{i+1}$, that avoids the neighbors of $e_{i+k+1}$ (mod $2k+1$).

3. Characterizations

Our first characterization of 2-directional orthogonal ray graphs is as follows.

Theorem 1 A bipartite graph $G$ is a 2-directional orthogonal ray graph if and only if a bipartite adjacency matrix of $G$ is $\gamma$-freeable, where

$\gamma = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, \quad \square$

Theorem 1 leads to the following characterization based on vertex ordering.

Corollary 1 A bipartite graph $G$ is a two-directional orthogonal ray graph if and only if $G$ is weakly orderable. \quad \square$

The following is a characterization by circular arc graphs.

Theorem 2 A bipartite graph $G$ is a two-directional orthogonal ray graph if and only if the complement of $G$ is a circular arc graph. \quad \square$

Combining Theorem 2 with a result by Feder, Hell, and Huang[1], we obtain the following characterization.

Corollary 2 A bipartite graph $G$ is a 2-directional orthogonal ray graph if and only if $G$ is chordal bipartite and contains no edge-asteroids. \quad \square$

Theorem 2 leads to some interesting consequences as follows. Since McConnell[5] showed a linear-time recognition algorithm for circular arc graphs, we have the following.

Theorem 3 It can be decided in $O(n^2)$ time whether an $n$-vertex graph is a 2-directional orthogonal ray graph. \quad \square$

From Theorems 1 and 3, we have the following theorem which settles the open problem of recognizing $\gamma$-freeable matrices (Problem 1 in [3]).
Theorem 4  It can be decided in $O((m+n)^2)$ time whether an $m \times n$ matrix is $\gamma$-freeable. □

Since Hsu [2] showed that graph isomorphism can be solved in $O(mn)$ time for $n$-vertex $m$-edge circular arc graphs, we have the following.

Corollary 3  The graph isomorphism problem can be solved in $O(n^3)$ time for $n$-vertex 2-directional orthogonal ray graphs. □

On the other hand, Uehara, Toda, and Nagoya [10] showed that the isomorphism problem is GI-complete for chordal bipartite graphs. Thus the class of 2-directional orthogonal ray graphs provides a boundary case for the complexity of graph isomorphism. This is an improvement from the earlier boundary class, the interval bigraphs, which is a proper subset of the class of 2-directional orthogonal ray graphs, as we shall show in next section.

4. Relation to Other Graph Classes

A bipartite graph $G$ with a bipartition $(U, V)$ is called a grid intersection graph if there exist a family of non-intersecting line segments $L_u, u \in U$, parallel to the $x$-axis in the $xy$-plane, and a family of non-intersecting line segments $L_v, v \in V$, parallel to the $y$-axis such that for any $u \in U$ and $v \in V$, $(u, v) \in E(G)$ if and only if $L_u$ and $L_v$ intersect.

A grid intersection graph is said to be unit if all the line segments corresponding to the vertices have the same length.

A bipartite graph $G$ with a bipartition $(U, V)$ is called an interval bigraph if every vertex $w \in U \cup V$ can be assigned an interval $I_w$ on the real line so that for all $u \in U$ and $v \in V$, $(u, v) \in E(G)$ if and only if $I_u$ and $I_v$ intersect.

The following observation is implicit in [4], and can be seen without difficulty.

Observation 1  A cycle $C_n$ of length $2n$ is an orthogonal ray graph if and only if $2 \leq n \leq 6$. □

We can show the following.

Observation 2  The class of orthogonal ray graphs is a proper subset of the class of unit grid intersection graphs. □

From Observation 1 and Corollary 2, we have the following.

Observation 3  The class of 2-directional orthogonal ray graphs is a proper subset of the class of orthogonal ray graphs. □

Otachi, Okamoto, and Yamazaki [6] showed that the class of graphs which have a $\gamma$-freeable bipartite adjacency matrix properly contains the class of interval bigraphs, and therefore we have the following.

Observation 4  The class of interval bigraphs is a proper subset of the class of 2-directional orthogonal ray graphs. □

The relationship between the various graph classes mentioned in this paper can be summarized as shown in Figure 1.

We conclude by noting that characterization and recognition of orthogonal ray graphs remain open.

References