On the Geodesic Diameter in Polygonal Domains*

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A polygonal domain $\mathcal{P}$ with n corners V and h holes is a connected polygonal region of genus h whose boundary consists of $h+1$ closed chains of n total line segments. The holes and the outer boundary of $\mathcal{P}$ are regarded as *obstacles*. Then, the geodesic distance $d(p,q)$ between any two points p, q in polygonal domain $\mathcal{P}$ is defined to be the (Euclidean) length of a shortest obstacle-avoiding path between p and q.

In this paper, we address the *geodesic diameter* problem. The geodesic diameter $\mathrm{diam}_{\mathcal{P}}$ of a given polygonal domain $\mathcal{P}$ is defined as follows:

$$\mathrm{diam}_{\mathcal{P}} := \max_{p,q\in\mathcal{P}} d(p,q).$$

The geodesic diameter problem is to find the value of $\mathrm{diam}_{\mathcal{P}}$ and a pair of points in the domain that realize the geodesic diameter.

The geodesic diameter problem can be seen also as a maximization problem: letting $f(p) := \max_q d(p,q)$ be the objective function, the problem is to find an optimal solution $p \in \mathcal{P}$ that maximizes $f(p)$. As variations, we can restrict the region $A \subseteq \mathcal{P}$ where p and q may span over. We write $\mathrm{diam}_{\mathcal{P}}(A) := \max_{p,q\in A} d(p,q)$; thus, $\mathrm{diam}_{\mathcal{P}} = \mathrm{diam}_{\mathcal{P}}(\mathcal{P})$.

When there is no hole inside $\mathcal{P}$ ($h = 0$), a surprising linear time algorithm is known by Hershberger and Suri [1], using a way of fast matrix searching. More surprisingly, however, there is no known algorithm on the geodesic diameter problem for general polygonal domains having one or more holes. A well-known open problem posed by Mitchell [3] asks an efficient algorithm for computing the corner-to-corner diameter $\mathrm{diam}_{\mathcal{P}}(V)$. Even in this special case, only known is a brute-force algorithm which takes $O(n^2 \log n)$ time, checking all geodesic distances between every pair of corners.

This fairly large gap between simple polygons ($h = 0$) and polygonal domains ($h \geq 1$) is seemingly due to the uniqueness of the shortest (geodesic) path between any two points. In a simple polygon, there is a unique shortest path between any two points inside it but this is not the case for general polygonal domains with holes. Using this uniqueness, one can show that the diameter $\mathrm{diam}_{\mathcal{P}}$ is indeed realized by a pair of corners when $h = 0$; that is, $\mathrm{diam}_{\mathcal{P}}(\mathcal{P}) = \mathrm{diam}_{\mathcal{P}}(V)$. On the other hand, we exhibit examples using the non-uniqueness of shortest paths when $h \geq 1$, which show that the diameter may be realized by non-corner points on $\partial\mathcal{P}$ or even by interior points of $\mathcal{P}$. See Figure 1. This observation implies an immediate difficulty in devising any straightforward algorithm since the search space like $\partial\mathcal{P}$ or the whole domain $\mathcal{P}$ is not finite.

The aim of this work is to find a first and efficient algorithm that computes $\mathrm{diam}_{\mathcal{P}}$ and a diametral pair in a given polygonal domain $\mathcal{P}$.

Observations and Results

We overcome this difficulty using further characterization of such counter-intuitive cases. Following is our characterization for the most difficult case (in a computational point of view), where the diameter $\mathrm{diam}_{\mathcal{P}}$ is determined by two interior points, in terms of the cells of shortest path maps $\mathsf{SPM}(u)$ for corners $u \in V$.

Observation 1 *Let $(s,t) \in \mathcal{P}\times\mathcal{P}$ be a diametral pair of $\mathcal{P}$. If both s and t are interior points of $\mathcal{P}$, then there are 6 distinct corners $u_1, u_2, u_3, v_1, v_2, v_3 \in V$ such that the following hold:*

- $d(s,t) = |su_i| + d(u_i, v_i) + |v_i t|$ *for* $i = 1, 2, 3$.
- $s \in (\mathrm{int}\triangle u_1u_2u_3) \cap \sigma_{v_1}(u_1) \cap \sigma_{v_2}(u_2) \cap \sigma_{v_3}(u_3)$.
- $t \in (\mathrm{int}\triangle v_1v_2v_3) \cap \sigma_{u_1}(v_1) \cap \sigma_{u_2}(v_2) \cap \sigma_{u_3}(v_3)$.

Above, $\triangle abc$ denotes the triangle of vertices a, b, c, $\mathrm{int}A$ denotes the interior of a set $A \subseteq \mathcal{P}$, and $\sigma_u(v)$

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*Work by S.W. Bae was supported by the Brain Korea 21 Project. Work by Y. Okamoto was supported by Global COE Program "Computationism as a Foundation for the Sciences" and Grant-in-Aid for Scientific Research from Ministry of Education, Science and Culture, Japan, and Japan Society for the Promotion of Science.

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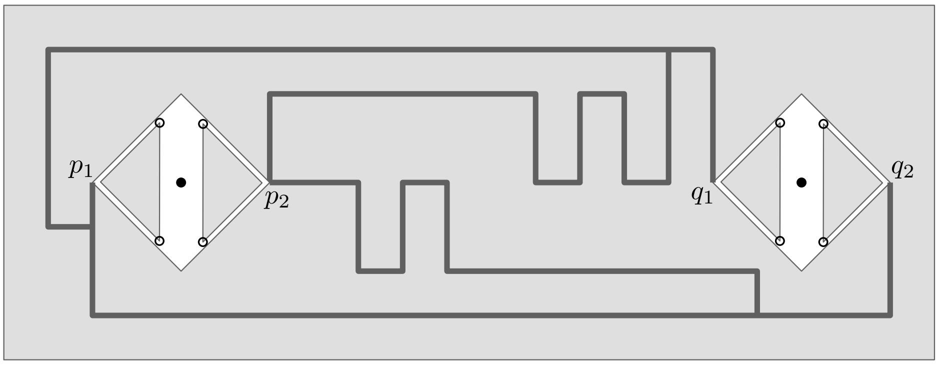

Figure 1: A polygonal domain $\mathcal{P}$ whose diameter is determined by two interior points: The main feature of this instance is a pair of square-shaped free space with two triangular holes inside each; gray shaded regions represent holes. These two squares are connected by four very narrow corridors (thick gray paths) between p_i and q_j for $i = 1, 2$ and $j = 1, 2$. Observe that $d(p_1, q_1) = d(p_1, q_2) = d(p_2, q_1) = d(p_2, q_2)$, and hence that the geodesic diameter is determined by the centers of the squares (two black dots), both of which are interior points of $\mathcal{P}$. In this example, eight corners (small circles) define the diametral pair and there are 16 topologically different shortest paths between the diametral pair, while Observation 1 assures six corners and the presence of three shortest paths.

denotes (the closure of) a cell of $\mathsf{SPM}(u)$ *with root* v.

The shortest path map $\mathsf{SPM}(u)$ for a fixed source $u \in \mathcal{P}$ is a decomposition of $\mathcal{P}$ into cells $\sigma_u(v)$ for $v \in V$ where any two points $a, b \in \sigma_u(v)$ have the topologically same shortest paths from u and their last corner is commonly v. $\mathsf{SPM}(u)$ has at most linear complexity and can be computed in time $O(n \log n)$ [2]. For more details, we refer to a survey article by Mitchell [3].

Based on Observation 1, one can design an exhaustive algorithm, checking all 6-tuples of corners V; this approach would yield an $O(n^{8+\epsilon})$-time algorithm. Our algorithm takes any triple (u_1, u_2, u_3) of corners and compute the overlay of the three shortest path maps $\mathsf{SPM}(u_i)$ in $O(n^2 \log n)$ time. Then, each cell σ of the overlay gives us another triple (v_1, v_2, v_3) since $\sigma = \bigcap \sigma_{u_i}(v_i)$ for some $v_i \in V$. Also, since there are at most $O(n^2)$ cells in the overlay, we are done by testing at most $O(n^5)$ 6-tuples of corners in V. We finally achieve a sub-n^6-time algorithm.

Theorem 1 *For a given polygonal domain* $\mathcal{P}$ *of* n *corners, the diameter* $\mathrm{diam}_{\mathcal{P}}$ *and a diametral pair of* $\mathcal{P}$ *can be computed in* $O(n^{5+\frac{10}{11}+\epsilon})$ *time for any arbitrarily small* $\epsilon > 0$.

Also, we consider a special case where there is a single hole ($h = 1$) in $\mathcal{P}$.

Observation 2 *If* $h = 1$*, then the diameter is always determined by two boundary points on* $\partial\mathcal{P}$.

This leads to a better time bound in this case.

Theorem 2 *If* $\mathcal{P}$ *contains only a single hole, the diameter and a diametral pair of* $\mathcal{P}$ *can be computed in* $O(n^{\frac{11}{3}} \log n)$ *time.*

Open Questions

We know that if the diameter is determined by two interior points, then there are at least three distinct shortest paths between them (Observation 1). The currently known example, however, shows 16 distinct shortest paths (Figure 1). Can one tighten the gap between the bounds? Also, can one find a more precise characterization to the cases where two interior points determine the diameter?

References

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