Maximum Eigenvalue Problem for Escherization

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A tiling of the plane is a collection of figures, called tiles, that cover the plane without gaps or overlaps except at their boundaries. Tilings have been used for various purposes, such as ornamentation for construction and design of clothes. The patterns made by tilings range from those made by simple geometrical figures to those made by intricate figures, and they attract a great deal of interest from both artists and mathematicians. From a mathematical point of view, the placement rules, incidence types, and other properties of tilings have been considered deeply [1].

The Dutch woodblock artist M. C. Escher is one of the greatest artists of artistic tilings. His works include many tilings made of animal forms based on his own trial-and-error approaches.

Kaplan and Salesin [3] first introduced the problem of finding tilings automatically in which the tiles are similar to a given shape. This problem is called the Escherization problem, named after Escher and his elegant work. More precisely, the Escherization problem is as follows.

> **Escherization problem** Given a closed plane figure S (the "original shape"), find a closed figure T such that:
>
> 1. T is as close as possible to S;
> 2. copies of T fit together to form a tiling of the plane.

Kaplan and Salesin proposed a solution to this problem using simulated annealing and concluded that their system performed well on convex or nearly convex shapes, but did not work well on intricate shapes. In addition, it seems that their system requires much time in computation because of the use of simulated annealing. Kaplan and Salesin also published a paper about dihedral Escherization that dealt with a tiling made from two tiles [2].

We consider the Escherization problem with a different approach. By reformulating the Escherization problem as a more tractable optimization problem, we propose a new method that is efficient and is able to deal with more intricate figures. First, we introduce a shape metric that is used as the criterion for how closely two shapes resemble each other, and it becomes the objective function of our optimization problem. We use the metric proposed by Werman and Weinshall [4] for this purpose. Next, we consider the constraints that should be satisfied by the tiles. Finally, we optimize the objective function under these constraints. The product is the optimal figure that fulfills the two conditions of the Escherization problem. Moreover, we transform this optimization problem into the problem of finding the maximum eigenvalue of a symmetric matrix. Then, we show that the optimal eigenvector corresponds to an orthogonal projection of the vector that represents the original shape onto the "space of tilable shapes". Thus, the task resolves to computing the orthogonal projection of the original shape vector, and this is straightforward.

We implemented our method as a computer-aided system for Escherization. Here we show examples of the behavior of our system. Figure 1 shows the Escherization of a "lizard". Panel (a) shows the pair of the input and output; the larger figure is the goal shape representing a lizard, and the smaller shape is the tile obtained as the optimized shape. Panel (b) shows the tiling generated by this tile. In this example, the system chose the tiling type associated with the symmetry group generated by rotation by 90 degrees and a translation.

Another example is shown in Figure 2, in which panel (a) shows the goal shape representing an octopus (the larger figure) and the tile obtained by the optimization (the smaller figure), and panel (b) shows the resulting tiling. It is the same type as that in Figure 1.

These examples show that our system can deal with complicated figures such as animals with many limbs.

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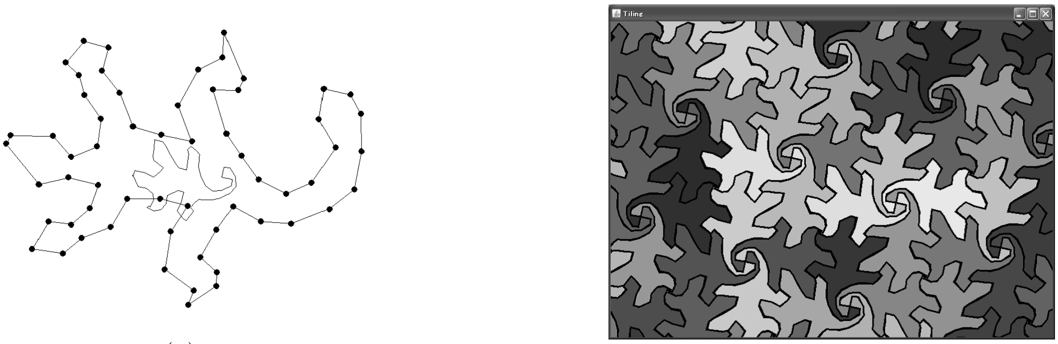

(a) (b)

Figure 1. Escherization of a lizard

(a) (b)

Figure 2. Escherization of an octopus

In our current system, 10 types of tiling patterns are implemented. Hence, for each input figure, the system solves 10 optimization problems and chooses the best one according to the shape metric. The total time for the computation is about 30 s by a standard notebook PC with a 2.0 GHz cpu and 2 GB RAM.

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References

[1] B. Grunbaum and G. C. Shephard. *Tilings and Patterns.* W. H. Freeman, New York, 1987.

[2] C. S. Kaplan and D. H. Salesin. Dihedral Escherization. *Proceedings of Graphics Interface,* London, Ontario, Canada, May 2004, pp. 255–262.

[3] C. S. Kaplan and D. H. Salesin. Escherization. *Proceedings of SIGGRAPH,* New Orleans, LA, July 2000, pp. 499–510.

[4] M. Werman and D. Weinshall. Similarity and Affine Invariant Distances Between 2D Point Sets. *IEEE Transactions on Pattern Analysis and Machine Intelligence,* Vol. 17 (1995), pp. 810–814.