Fixed Center of Mass Configurations of Three Points in Given Curves

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We consider a configuration of three points q_1, q_2 and q_3 in a given planer curve γ whose center of mass is at the origin, i.e., $q_1 + q_2 + q_3 = 0$, $q_i \in \gamma$, $i = 1, 2, 3$. This configuration is important in classical mechanics because the center of mass is always conserved when the three points move according to the Newton's laws of motion. Therefore if the three points move in the γ and if this configuration is unique, they always take this configuration.

We found the following theorem [1].

Theorem 1. *If a curve γ is centrally symmetric then the set $\{\{q_1, q_2\} | q_1, q_2 \in \gamma, q_1 + q_2 + q_3 = 0\}$ for a given q_3 is equal to the set $\{\{q, q^*\} | q \in \gamma \cap \gamma_{\|}\}$ where $\gamma_{\|}$ is the parallel translation $q \mapsto q - q_3$ of the curve γ and $q^* = -q - q_3$.*

See figure 1. Extensions and proofs of theorem 1 is given below.

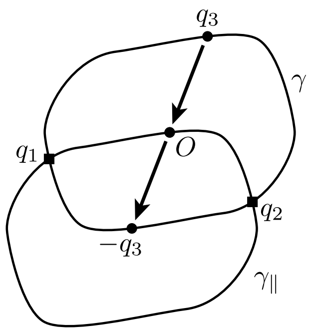

Fig. 1. The curve γ is centrally symmetric with respect to the origin O. For a given point $q_3 \in \gamma$, the two points on γ that satisfy $q_1 + q_2 + q_3 = 0$ are given by the cross points of γ and $\gamma_{\|}$.

Using theorem 1 and the number of cross points between closed convex curve and its translation, we proved that if the curve γ is eight shaped with some conditions then the pair $\{q_1, q_2\}$ which satisfies $q_1 + q_2 + q_3 = 0$ for a given q_3 is unique, where $q_i \in \gamma$ and $q_i \neq 0$ for $i = 1, 2, 3$ [1].

The three-point motion in the eight shaped orbit was found in 1993 [2] and its existence was proved independently in 2000 [3]. However the shape of the orbit is not known. We think the uniqueness of the configuration for the eight shaped curve may play an important role in the study of the shape of the orbit.

In fact, we found another geometric condition for this motion. The tangent lines from the three points with this configuration have to meet at a point[4]. This condition is a more strict constraint for the shape of the orbit and would be a clue to find it.

At the conference, we shall explain the outline of the above results and will give the related open problems which may be solved by the researchers in the area of the conference.

Extensions and proofs of theorem 1

Theorem 1 is a corollary of the following more general theorem [1].

Theorem 2. *For a given set* $\gamma_1, \gamma_2 \subset \mathbb{R}^d$ *and* $q_3 \in \mathbb{R}^d$*, we have the following equalities:*

$$\{\{q_1, q_2\} | q_1 \in \gamma_1, q_2 \in \gamma_2, q_1 + q_2 + q_3 = 0\} = \{\{q_1, q_1^*\} | q_1 \in \gamma_1 \cap \gamma_2^*\} \tag{1}$$
$$= \{\{q_2, q_2^*\} | q_2 \in \gamma_1^* \cap \gamma_2\}, \tag{2}$$

where * *represents the inversion with respect to the point* $-q_3/2$.

Proof: We prove equation (1). Equation (2) is proved similarly. If q_1 and q_2 satisfy $q_1 \in \gamma_1$, $q_2 \in \gamma_2$ and $q_1 + q_2 + q_3 = 0$, then $q_1 = -q_2 - q_3 = q_2^* \in \gamma_2^*$, since the inversion with respect to the point $-q_3/2$, $q^* = -(q + q_3/2) - q_3/2$ is represented by $q^* = -q - q_3$. Therefore, $q_1 \in \gamma_1 \cap \gamma_2^*$ and $q_2 = q_1^*$. Inversely, if q_1 and q_2 are given by $q_1 \in \gamma_1 \cap \gamma_2^*$ and $q_2 = q_1^*$, then $q_1 \in \gamma_1 \cap \gamma_2^* \subset \gamma_1$, $q_2 = q_1^* \in \gamma_1^* \cap \gamma_2 \subset \gamma_2$. Moreover, by the definition of $q_2 = q_1^*$ we get $q_1 + q_2 + q_3 = 0$. □

If q_1, q_2 and q_3 move around the same set γ, we have the following corollary by simply making $\gamma_1 = \gamma_2 = \gamma$.

Corollary 1. *For a given set* $\gamma \subset \mathbb{R}^d$ *and* $q_3 \in \gamma$*, we have the following equality:*

$$\{\{q_1, q_2\} | q_1, q_2 \in \gamma, q_1 + q_2 + q_3 = 0\} = \{\{q, q^*\} | q \in \gamma \cap \gamma^*\}. \tag{3}$$

We do not assume any symmetry for the set γ in this corollary. So, this corollary can be used to make fixed center of mass configurations of three points in given curves with no symmetry.

Note that the inversion with respect to the point $-q_3/2$, $q \mapsto q^*$ can be decomposed into the inversion with respect to the origin followed by the parallel translation $q \mapsto q - q_3$. Therefore, the curve γ^* can be made by the two steps. First make inversion γ' of γ with respect to the origin, then make parallel translation $\gamma'_{||}$ of γ' by $q \mapsto q - q_3$. For the case γ is invariant under the inversion with respect to the origin, then $\gamma' = \gamma$ and $\gamma^* = \gamma_{||}$. Thus, we get a proof of theorem 1.

References

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