CONVEX SETS IN A REAL PROJECTIVE SPACE AND ITS APPLICATION TO COMPUTATIONAL GEOMETRY

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In a real projective space P, a pair of distinct points determines a unique projective line passing through these points. However, there are two segments continuously joining these points because a real projective line is homeomorphic to a circle. A natural question arises how we can introduce a notion of convexity in P. The first introduction probably dates back to Steinitz[1], Veblen and Young[5]. For general references of convex geometry, see [3][4][6]. The following definition is equivalent to the one by Steinitz[2].

Definition 1. *A subset S of P is simple convex if for any two points of S, exactly one of the segments determined by these points is contained in S.*

Since the intersection of simple convex sets may fail to be simple convex, we define our notion of convexity as follows.

Definition 2. *A subset of P is convex if it is expressed as the intersection of a nonempty family of simple convex sets. The family of all convex sets in P which is ordered by inclusion is written as $\mathcal{C}(P)$.*

In this setting, simple convex sets are exactly connected convex sets. Each connected component of a convex set is a simple convex set. Let us introduce a subclass of convex sets in P, which has somewhat a topological flavor.

Definition 3. *A convex set in P is saturated if it is expressed as the intersection of a nonempty family of open simple convex sets. The family of all nonempty saturated convex sets in P which is ordered by inclusion is written as $\mathcal{C}^{\circ}_{\mathrm{sat}}(P)$.*

Since there are nonsaturated convex sets, saturated convex sets form a proper subclass of convex sets. For any convex set, there is the least saturated convex set containing it.

In this talk, we concentrate on saturated convex sets. One notable property is that for a dual projective space P^*, there is a one-to-one correspondence between $\mathcal{C}^{\circ}_{\mathrm{sat}}(P)$ and $\mathcal{C}^{\circ}_{\mathrm{sat}}(P^*)$, which is considered as an extension of the projective duality between points and hyperplanes.

Theorem 1. *Let P be a real projective space and let P^* be a dual projective space of P. Then the mapping $\Phi : \mathcal{C}^\circ_{\mathrm{sat}}(P) \to \mathcal{C}^\circ_{\mathrm{sat}}(P^*)$,*

$$\Phi(C) := \{\delta(h) \in P^* | h \text{ is a hyperplane in } P \text{ which avoids } C\}$$

is an anti-order isomorphism, where δ denote a dual correspondence between hyperplanes in P and points in P^*.

This theorem implies that in a real projective space, a notion of *linearly convex sets*, which is yet another projective convexity [7], is almost equivalent to a notion of saturated convex sets. The only difference is that the ground set P is a linearly convex but not saturated convex set.

We give an application of the projective convexity toward computational geometry. Let us consider the following classification problem of a family of sets in $\mathbb{R}^2$ by a line.

Problem 1. *Let $S_1, ..., S_n$ be nonempty finite subsets of $\mathbb{R}^2$ each of which is called a class. Given two distinct classes S_i and S_j, decide whether there is a line such that it separates S_i and S_j, and moreover for any other class S_k $(k \notin \{i, j\})$, it does not separate S_k into two nonempty subsets.*

This problem in case $k = 2$ is well-known as the problem of linearly separability and can be reduced to the problem of the intersection of two convex hulls $[S_1]$ and $[S_2]$. Although this technique is not available for $k > 2$, we can transform it into the problem of the intersection of convex sets in a projective plane in the following way.

Since $\mathbb{R}^2$ can be embedded in a projective plane P^2, each convex hull $[S_k]$ in $\mathbb{R}^2$ for $k \leq n$ is a simple convex set in P^2. Thus we can make use of the dual mapping Φ in Theorem 1 by which for $k \leq n$, the family of all lines avoiding $[S_k]$ is transformed into a nonempty saturated convex set $\Phi([S_k])$ in a dual plane. Notice that for any point in $\bigcap_{k\leq n} \Phi([S_k])$, the corresponding line in P^2 avoids all members $[S_k]$ $(k \leq n)$. Therefore for distinct two classes S_i and S_j, it suffices to decide whether there is a point in $\bigcap_{k\leq n} \Phi([S_k])$ such that the corresponding line separates S_i and S_j in $\mathbb{R}^2$.

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