Regions of Empty Overlapping Circles

(Extended Abstract)

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I. Introduction

The well-known Voronoi diagram [1] is one of the most useful data structures in computational geometry. It helps to solve problems, allowing the design of simple and efficient algorithms whenever proximity informations are needed. In particular it helps solving the following problem:

Let S be a set of two or more points in the plane. For convenience, elements of S are called *sites*. A "query-point" x, being distinct from elements of S is given. A question may rise : What is the largest *open disc* centered on x, containing no site of S in its interior ? Obviously it is equivalent to ask: What is the largest empty *circle* passing through one or more site of S ? The Voronoi regions *encode the precise required information*. So using the Voronoi partition, the problem is trivial, we simply check in which Voronoi region is x.

Now let us consider a slightly different problem. Again, S is a set of two or more sites and a "query-point" x which belongs not to S is chosen. Given an *arbitrary configuration of one or more overlapping circles* (see Fig.1a for instance), what is the maximal size it can take (when located on x) such that its interior remains empty of S ?

Considering the fact there exists a nice geometric structure, the Voronoi diagram, for solving the first problem elegantly (paragraph 2). There may exist a variant of this diagram for solving the second one ? A new theoretical structure has been designed for this purpose and is presented in this paper.

While it has surprizing properties, e.g. non-covering of the whole plane (so it is not a partition in the general case), it is actually a generalized Voronoi diagram. We think the structure is worth of interest because applications might be numerous, e.g. robotics path planning problems, patterns detection, etc.

The structure could also be a *unifying approach* for state-of-the-art problems. For instance: Polygon placement among points or obstacles [2], [3]. For finding empty instances of particular shapes, Voronoi regions can be computed normally but using a different metric. This way, Lee [4] was able to store empty occurences of squares. Numerous "generalized Voronoi diagrams" use a non-Euclidean metric, therefore it is possible to store locations of various convex empty shapes. (the unit-ball determines the shape) as explained in [5]. Yet, these approaches are not satisfying. It does not enable the use of *any shape*, for instance: non-convex shapes, shapes with holes, etc.

II. Preliminaries

The *base objects* of our work; so-called shape-models, are introduced in this section. A *shape-model* express more precisely and rigorously what means "arbitrary configuration of overlapping circles". Thanks to these, a mathematical definition of *Regions of Point-free Shape-models* can be given in section III.

A. Notations

$\boldsymbol{b}(x,r)$ is the *open disc* of radius r centered on point x. Its boundary is a *circle*, denoted $\boldsymbol{c}(x,r)$. Mathematically:

$$\boldsymbol{b}(x,r) = \left\{ y \in \mathbb{R}^2 \;\middle|\; \|y-x\| < r \right\}$$
$$\boldsymbol{c}(x,r) = \left\{ y \in \mathbb{R}^2 \;\middle|\; \|y-x\| = r \right\}$$

where $\|.\|$ denotes the Euclidean metric.

B. Shape-models definition

We define a shape-model as a set of circles, such that one of these is designated as being the *reference circle*. In order to have simple notations, ordered lists are used and the reference disc is the first one by agreement.

Formally, let $M = (C, R)$ be a shape-model constituted of m circles. The m-tuple $C = (c_1, c_2, \ldots, c_m)$ represents the centers of circles. Elements of $R = (r_1, r_2, \ldots, r_m)$ represent the associated radii (real numbers). Moreover, M must satisfy the "overlapping circles property" if m is superior to 1 :

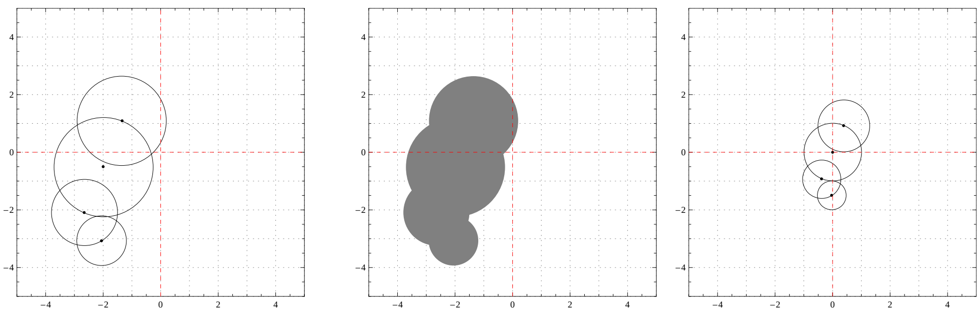
$$\forall 1 \leq i \leq m, \exists j \neq i \text{ such that } \boldsymbol{b}(c_i, r_i) \cap \boldsymbol{b}(c_j, r_j) \neq \emptyset$$

An example of a simple shape-model constituted of four circles is given in Fig. 1a.

C. Handling operations

Two operations are defined on shape-models:

- The $\boldsymbol{s}$ operation (for synthesis) builds a 2D region from a shape-model as shown on Fig. 1b. The function simply considers open discs contained in circles composing the shape-model, and computes the union of these discs.
- The $\boldsymbol{th}$ operation (for translation-homothety) takes three arguments: a shape-model $M = (C, R)$ with $C = (c_1, \ldots, c_m)$ and $R = (r_1, \ldots, r_m)$ as said previously, a point p and a scalar λ.
  It translates every circle of M by a vector $\overrightarrow{c_1 p}$ and rescales them by a factor (λ / r_1). Hence, the resulting shape-model is noted $\boldsymbol{th}(M, p, \lambda)$. An example is given in Fig. 1c.



(a) shape-model M_0 (b) $\boldsymbol{s}(M_0)$ (c) $\boldsymbol{th}(M_0, (0,0), 1)$

Figure 1: The synthesis operation, denoted $\boldsymbol{s}$ and the translation-homothety operation, denoted $\boldsymbol{th}$

III. Regions of Empty Shape-models

The preliminary work allows geometric problems concerning sites and shape-models to be examined algebraically. Algebraic representation let us define the new structure and compute it. Given a finite set of sites S and given any shape-model M. For each $p \in S$ the *Region of Point-free Shape-models* generated by $p \in S$ is defined by:

$$R_M(p) = \left\{ x \in \mathbb{R}^2 \mid \boldsymbol{s}\left(\boldsymbol{th}(M, x, \|x - p\|)\right) \cap S = \emptyset \right\}$$

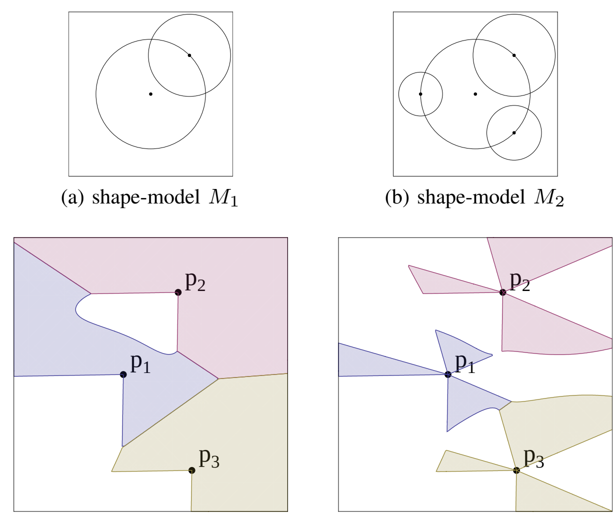
Intuitively, $R_M(p)$ represents the locations $x \in \mathbb{R}^2$ where M can be translated (ie. its reference circle center becomes x) *and expanded until it has* p *on its boundary*, while remaining empty of sites. Thus, a *Region of Point-free Shape-models* is defined for a given shape-model M (noted in subscript) and a site p as an argument; global notation $R_M(p)$.

IV. Results

In order to ease the implementation, the structure has been decomposed (using the definitions of region and handling operations) into $m \times n$ sub-regions of the plane (m being the number of discs, n the number of sites in S). Each sub-region is defined algebraically by a single inequation. Therefore a large system of inequations can encode the *Regions of Point-free Shape-models* structure. The computation of region boundaries relies on the method of resultants [6]. Finally a hybrid implementation has been realized based in part on *C software* (using *GNU Scientific Library* [7] for polynomial calculus and representation). Then, the computational software *Mathematica* is automatically used for high-level regions encoding and plotting. Communication is made through *C/C++ MathLink interface* [8], [9].

An example of the produced results is shown on Fig. 2. The computation is fast even with latence due to the interfacing. Further work will focus on the realization of a stand-alone program in order to enhance the execution speed and to study full algorithm complexity.

Note that the shape-models are not limited except by the overlapping property. A shape-model could contain holes for example.



(a) shape-model M_1 (b) shape-model M_2

(c) $R_{M_1}(S)$ (d) $R_{M_2}(S)$

Figure 2: Respective "Regions of Empty Shape-models" given two different shape-models, and a same 3-site set S. Colors are used to differentiate two regions associated to different sites of S $R(p)$ and $R(p')$, p, p' two different elements of S.

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