

Locally Density Adaptive Alpha Shape

-Extended Abstract-

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1. Introduction

This talk deals with the problem of reconstructing shapes from an unorganized set of sample points (called S). First, we give an intuitive notion for gathering sample points in order to reconstruct a shape. Then, we introduce a new generalization of the convex hull inspired by α -shape [1] but which takes into account the density variation of the sample points. This is called “*Locally-density-adaptive- α -shapes*” (*LDA- α -shapes*).

Our work is quite similar to the “conformal- α -shapes” [2] and to the “automatic surface reconstruction with α -shape method” [3]. The main difference is that LDA- α -shapes are intended to reconstruct surface in \mathbb{R}^2 or volume in \mathbb{R}^3 , or to do clustering in higher dimension. So, for instance, to reconstruct a surface in \mathbb{R}^2 , the considered set of sample points is not just lying on the boundary of the shape but covers the whole area of the shape and its density varies from place to place, depending on the required amount of details.

At the time, the LDA- α -shape is defined in \mathbb{R}^2 , but it will be easily generalized in higher dimension. The LDA- α -hull is formally defined and some nice properties are proven. It generates a monotone family of hulls for α ranging from 0 to 1. Afterwards, from LDA- α -hull, we formally define, on the one hand, the LDA- α -shape describing the boundaries of the reconstructed shape and, on the other hand, the LDA- α -complex describing the shape and its interior. These definitions lead to a very simple and efficient algorithm to compute LDA- α -shape and LDA- α -complex in $O(n \log(n))$.

2. Related geometrical concept

2.1. General position assumption

In the following, we consider a set of points $S \subset \mathbb{R}^2$ and we assume that points of S are in general position, that is, there are neither three points on a line nor four points on a circle.

2.2. Delaunay Triangulation

The Delaunay Triangulation of S is the only triangulation whose the circumcircle of each triangle does not contains any point of S in its interior. In the following, the Delaunay triangulation of S is denoted $Del(S)$.

2.3. Maximal disk

Definition 1: We define a *maximal disk* of S either, an open disk containing no point of S and whose boundary passes through three points of S , or, an open half plane containing no point of S and whose boundary passes through two points of S .

In the first case, a maximal disk is the interior of the circumcircle of a Delaunay triangle and is also called a *finite maximal disk*. In the second case, a maximal disk is an open half plane containing no point of S and bounded by the straight line spanned by a convex hull edge. It is also called an *infinite maximal disk* and its radius is considered to be infinite.

3. Sample condition and idea of reconstruction

In order to reconstruct a region Σ , its set of sample points S has to carry enough information. Some assumptions about its quality have to be made. First, points of S are all supposed included in the interior of Σ . While the sample is not regular, it is nevertheless under conditions. In particular, it has to be of strong density when a big amount of detail is required.

Our idea of reconstruction utilizes this concept. From the first assumption, points of S are inside Σ . It remains to find out the kind of areas between them. Which areas are inside the shape? Which areas are outside the shape? The problem comes down to determining how to efficiently obtain a significant measure of the local density of S for all points of S .

Correctly, by definition the maximal disks of S are empty. Thus, the larger they are, the sparser is the density at this place. Moreover, the ratio between two neighboring maximal disks gives a good interpretation of density variation between both places. The idea is to consider that a maximal disk relatively large compared with all its neighboring maximal disks means it is included outside the shape. Our reconstruction algorithm is based on this simple idea and its concept is formally explained in what follows.

4. Definitions

4.1. LDA- α -empty disk

Definition 2: let D be a maximal disk with radius r , P be the set of all points of S lying on the boundary of D and α a positive real number. The disk D is said to be *LDA- α -empty* if and only if, for all p of P , there exists a maximal disk with radius r' , whose boundary passes through p , and such that $\frac{r'}{\alpha} \leq r$.

Observation : all infinite maximal disks are LDA- α -empty, for all α since they are all infinitely larger than any finite maximal disk.

Property 1: There exists an α' such that, for every $\alpha \leq \alpha'$, the maximal infinite disks are the LDA- α -empty disks.

Proof: Let D be a finite maximal disk with radius r , let p be a point of S on the boundary of D , and D' be a maximal disk with radius r' whose boundary contains p . Let r_{max} be the radius of the largest finite maximal disk of S , r_{min} be the radius of the smallest one, and α such that $\frac{r_{min}}{r_{max}} > \alpha$. Since $\frac{r'}{\alpha} > r' / \frac{r_{min}}{r_{max}} = r' \frac{r_{max}}{r_{min}} \geq r_{min} \frac{r_{max}}{r_{min}} = r_{max} \geq r$, D is not LDA- α -empty.

Thus, we extend the range of α . It is a non-negative real number and we set that all infinite maximal disks and only infinite maximal disks are LDA-0-empty.

Property 2: For every $\alpha \geq 1$, all maximal disks are LDA- α -empty.

Proof: Let D be a maximal disk with radius r , p be a point of S on the boundary of D , D' be a maximal disk with radius r' whose boundary contains p , and α such that $\alpha \geq 1$. Now, choosing $D' = D$, we have $\frac{r'}{\alpha} = \frac{r}{\alpha} \leq r$ and thus D is LDA- α -empty.

From now on, α is ranging from 0 to 1.

4.2. LDA- α -hull

Definition 3: The LDA- α -hull of S is the intersection of all complements of the closures of LDA- α -empty disks.

Property 3: The LDA-0-hull of S is the convex hull of S .

Proof: From property 1, the infinite maximal disks are the LDA-0-empty disks. By definition, the LDA-0-hull is the intersection of all half-planes whose closures contain S and whose boundaries pass through two points of S . Hence, it is the convex hull of S .

Property 4: The LDA-1-hull of S is S .

Proof: The interiors of the circumcircles of the Delaunay triangles of S , i.e., the finite maximal disks cover the convex hull of S , but S since they are opens. The infinite maximal disks cover the complement of the convex hull of S , but S because S is included in the convex hull.

Observation : if $\alpha_1 \leq \alpha_2$ then a LDA- α_1 -empty disk is also LDA- α_2 -empty. Thus,

Property 5: If $\alpha_1 \leq \alpha_2$ then the LDA- α_1 -hull is a set containing the LDA- α_2 -hull.

Thus the LDA- α -hulls of S for α ranging from 0 to 1 form a discrete monotone family of hulls, from the convex hull of S to S itself.

4.3. LDA- α -shapes

Definition 4: The points of S lying on the boundary of LDA- α -empty disks are LDA- α -extreme.

From Property 5, we have:

Property 6: For $\alpha_1 \leq \alpha_2$, the set of all LDA- α_1 -extreme points is included in the set of all LDA- α_2 -extreme points.

Definition 5: Two points p and p' of S are LDA- α -neighbors if p and p' are LDA- α -extreme, both lie on the same LDA- α -empty disk, and the open edge pp' is included in a unique LDA- α -empty disk.

Now, LDA- α -shapes can be defined similarly to α -shapes [1].

Definition 6: Given a set S and a real number α ranging from 0 to 1, the LDA- α -shape of S is the straight line graph whose

vertices are the LDA- α -extreme points and whose edges connect the respective LDA- α -neighbors.

It immediately follows from properties 3 and 4 that a LDA-0-shape of S is the boundary of the convex hull of S and that a LDA-1-shape of S is S .

A LDA- α -shape of S is obviously a sub-graph of $Del(S)$ because every edge connects two points which are on the same maximal disk.

A LDA- α -shape is a straight line graph. It describes the boundary of the reconstructed domain. It is sometime necessary to distinguish the interior from the exterior: interior faces are those which contain the sample. Another way to describe the domain considering interior and exterior is done by defining LDA- α -complex.

4.4. LDA- α -complex

Definition 7: Given a set S and a real number α ranging from 0 to 1, the LDA- α -complex of S is the straight line graph whose edges are edges of $Del(S)$ included in at most one LDA- α -empty disk.

By definition, the LDA- α -shape of S is included in the LDA- α -complex of S .

It immediately follows that a LDA-0-complex of S is the Delaunay triangulation of S and a LDA-1-complex of S is S .

Moreover, LDA- α -complexes of S describes a discrete monotone family of sub-graphs of $Del(S)$ from $Del(S)$ to S , when α varies from 0 to 1.

5. Conclusion

We introduced the notion of LDA- α -hulls, LDA- α -shapes and LDA- α -complexes of a sample set of points of the plane. Like the α -hulls [1], from which we took inspiration, the LDA- α -hull proposes a generalization of the convex hull. Variation of α spawns a monotone family of hulls from the “crudest” to the “finest” interpretation of the sample. The main difference is that the variation of the density of the set of sample points is taken into consideration. A simple and efficient algorithm based on this concept exists.

Our future works will primarily consist in providing sample conditions of the shape in order to prove its reconstruction according to a range of α before to generalize this concept to the higher dimensions.

References

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