The acyclic chromatic number of a graph $G$, denoted $a(G)$, is the minimum number of colors required to properly color the vertices of a graph such that there are no bifichromatic cycles. The concept of acyclic coloring of a graph was introduced by [5] and is further studied in the last two decades in several works. Kostochka [6] proves that determining it is an NP-complete problem.

Given the computational difficulty involved in determining $a(G)$, several authors have looked at acyclically coloring particular families of graphs [5, 7, 3]. Using the probabilistic method, it was shown by Alon et al. [2] that any graph of maximum degree $\Delta$ can be acyclically colored using $O(\Delta^{4/3})$ colors. Focusing on the family of graphs with a small maximum degree $\Delta$, it was proved by Skulrattanakulchai [7] that $a(G) \leq 4$ for any graph of maximum degree 3. The work of Skulrattanakulchai was extended by Fertin and Raspaud [4] to show that it is possible to acyclically vertex color a graph $G$ of maximum degree $\Delta$ using at most $\Delta(\Delta + 1)/2$ colors. Recently, Yadav et al. [8] extended the work of Skulrattanakulchai [7] to show that any graph of maximum degree 5 can be colored using at most 8 colors. Burnstein [3] showed that $a(G) \leq 5$ for any graph of degree maximum 4. In this paper we prove the same result of [3] using 5 colors using a linear time algorithm.

Let $N(v)$ denote the set of neighbors of $v$, a partial coloring is an assignment of colors to a subset of $V(G)$ such that the colored vertices induce a graph with an acyclic coloring. Suppose $G$ has a partial coloring. Let $\alpha, \beta$ be any two colors. An alternating $\alpha, \beta$-path is a path in $G$ with each vertex colored either $\alpha$ or $\beta$. An alternating path is an alternating $\alpha, \beta$ path for some colors $\alpha, \beta$. A path is odd or even according to the parity of number of edges it contains. Let $v$ be an uncolored vertex. A color $\alpha \in [5]$ is available for $v$ if no neighbor of $v$ is colored $\alpha$. A color $\alpha \in [5]$ is feasible for $v$ if assigning color $\alpha$ to $v$ still results in a partial coloring.

We derive our result by extending a partial coloring by one vertex $v$ at a time. During this process, in some scenarios it is required that we recolor some of the vertices already colored so as to make a color feasible for the vertex which we try to color. However, note that this recoloring, if required, is limited to the neighborhood of the neighbors of $v$, in all cases. Thus, we show the following theorem, using Lemmata 1.2,1.3,1.4.

**Theorem 1.1** The vertices of any graph $G$ of degree at most 4 can be acyclically colored using five colors in $O(n)$ time, where $n$ is the number of vertices.

**Lemma 1.2** Let $\pi$ be any partial coloring of $G$ using colors in $[5]$ and let $v$ be any uncolored vertex. If $v$ has less than 3 colored neighbors, then there exists a color $\alpha \in [5]$ feasible for $v$.

**Lemma 1.3** Let $\pi$ be any partial coloring of $G$ using colors in $[5]$ and let $v$ be any uncolored vertex. If $v$ has exactly three colored neighbors, then there exists a partial coloring $\pi_1$ of $G$ using colors in $[5]$ and a color $\alpha \in [5]$ so that $\pi_1$ has the same domain as $\pi$, $\pi(x) \neq \pi_1(x)$ implies $x \in N(v)$ and $\alpha$ is feasible for $v$ under $\pi_1$. 

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Lemma 1.4 Let \( \pi \) be any partial coloring of \( G \) using colors in \([5]\) and let \( v \) be any uncolored vertex. If \( v \) has four colored neighbors, then there exists a partial coloring \( \pi_1 \) of \( G \) using colors in \([5]\) and a color \( \alpha \in [5] \) so that \( \pi_1 \) has the same domain as \( \pi \), \( \pi(x) \neq \pi_1(x) \) implies \( x \in N(v) \) or \( x \in N(N(v)) \), and \( \alpha \) is feasible for \( v \) under \( \pi_1 \).

Moreover, in all the above lemmata, both \( \pi_1 \) and \( \alpha \) can be found in \( O(1) \) time.

2 A Sketch of the Proofs of Lemmata 1.2, 1.3, and 1.4

\[ \text{Lemmata 1.2, 1.3:} \quad \text{Notice that when extending a partial coloring to a vertex} \ v, \ \text{if} \ v \ \text{has less than 3 colored neighbors, then there always exists a feasible color out of the five colors. In the case that} \ v \ \text{has three colored neighbors, then we may not have a feasible color when two of these neighbors have the same color. In this case, we recolor a neighbor of} \ v \ \text{that is a single vertex, if such a vertex exists. Otherwise, it can be shown that there always exists a feasible color for} \ v. \]

\[ \text{Lemma 1.4:} \quad \text{In the case that all the four neighbors of} \ v \ \text{are colored, it is more difficult to see which vertices have to be recolored so that a feasible color for} \ v \ \text{can be found. In this direction, we investigate the colors in the 2-neighborhood of} \ v. \ \text{Suppose that three neighbors of} \ v \ \text{have the same color but the other neighbor is different from these two. Assume without loss of generality that} \ \pi(w) = \pi(x) = \pi(y) = 1, \ \text{and} \ \pi(z) = 2. \ \text{Considering the number of possible dangerous} \ 1, \beta \ \text{dangerous cycles, there may be no feasible color for} \ v. \ \text{We consider two cases. When any of} \ \{w, x, y\} \ \text{is a single vertex, we recolor a single vertex from one of} \ w, x, y. \ \text{Depending on the new color of say, w.l.o.g,} \ w, \ \text{one can find a color that is feasible for} \ v. \ \text{When none of} \ \{w, x, y\} \ \text{are single vertices, then three} \ 1, \beta \ \text{dangerous cycles must exist, for otherwise there is a feasible color for} \ v. \ \text{This implies that} \ \{w, x, y\} \ \text{contain two differently colored neighbors. Assume, w.l.o.g, that} \ w \ \text{has neighbors colored so that color} \ 4 \ \text{appears at neighbors} \ w_1 \ \text{and} \ w_2 \ \text{and that color} \ 5 \ \text{appears at neighbor} \ w_3. \ \text{Then, we can recolor} \ w \ \text{if color} \ 2 \ \text{or} \ 3 \ \text{is missing in the neighbourhood of} \ w_1 \ \text{or} \ w_2. \ \text{Otherwise, it can be noticed that one of} \ w_1 \ \text{or} \ w_2 \ \text{should be a single vertex. This allows us to recolor} \ w_1 \ \text{from} \ 4 \ \text{to} \ 5. \ \text{This helps us in similarly exploring the colors in the neighborhood of} \ w_1 \ \text{and} \ w_3. \ \text{It then holds that either} \ w_3 \ \text{is also single or there exists a feasible color for} \ v. \ \text{The full report, dealing with all possible cases, is available as}[\text{[10]}]. \]

References


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