Acyclic Vertex Coloring of Graphs of Maximum Degree 4

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1 Indroduction

The acyclic chromatic number of a graph G, denoted $a(G)$, is the minimum number of colors required to properly color the vertices of a graph such that there are no bichromatic cycles. The concept of acyclic coloring of a graph was introduced by [5] and is further studied in the last two decades in several works. Kostochka [6] proves that determining it is an NP-complete problem.

Given the computational difficulty involved in determining $a(G)$, several authors have looked at acyclically coloring particular families of graphs [5, 7, 3]. Using the probabilistic method, it was shown by Alon et al. [2] that any graph of maximum degree Δ can be acyclically colored using $O(\Delta^{4/3})$ colors. Focusing on the family of graphs with a small maximum degree Δ, it was proved by Skulrattanakulchai [7] that $a(G) \leq 4$ for any graph of maximum degree 3. The work of Skulrattanakulchai was extended by Fertin and Raspaud [4] to show that it is possible to acyclically vertex color a graph G of maximum degree Δ using at most $\Delta(\Delta + 1)/2$ colors. Recently, Yadav et al. [8] extended the work of Skulrattanakulchai [7] to show that any graph of maximum degree 5 can be colored using at most 8 colors. Burnstein [3] showed that $a(G) \leq 5$ for any graph of degree maximum 4. In this paper we prove the same result of [3] using 5 colors using a linear time algorithm.

Let $N(v)$ denote the set of neighbors of v, a partial coloring is an assignment of colors to a subset of $V(G)$ such that the colored vertices induce a graph with an acyclic coloring. Suppose G has a partial coloring. Let α, β be any two colors. An alternating α, β-path is a path in G with each vertex colored either α or β. An alternating path is an alternating α, β path for some colors α, β. A path is odd or even according to the parity of number of edges it contains. Let v be an uncolored vertex. A color $\alpha \in [5]$ is *available* for v if no neighbor of v is colored α. A color $\alpha \in [5]$ is *feasible* for v if assigning color α to v still results in a partial coloring.

We derive our result by extending a partial coloring by one vertex v at a time. During this process, in some scenarios it is required that we recolor some of the vertices already colored so as to make a color feasible for the vertex which we try to color. However, note that this recoloring, if required, is limited to the neighborhood of the neighbors of v, in all cases. Thus, we show the following theorem, using Lemmata 1.2,1.3, and 1.4.

Theorem 1.1 *The vertices of any graph G of degree at most 4 can be acyclically colored using five colors in $O(n)$ time, where n is the number of vertices.*

Lemma 1.2 *Let π be any partial coloring of G using colors in $[5]$ and let v be any uncolored vertex. If v has less than 3 colored neighbors, then there exists a color $\alpha \in [5]$ feasible for v.*

Lemma 1.3 *Let π be any partial coloring of G using colors in $[5]$ and let v be any uncolored vertex. If v has exactly three colored neighbors, then there exists a partial coloring π_1 of G using colors in $[5]$ and a color $\alpha \in [5]$ so that π_1 has the same domain as π, $\pi(x) \neq \pi_1(x)$ implies $x \in N(v)$ and α is feasible for v under π_1.*

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Lemma 1.4 *Let π be any partial coloring of G using colors in $[5]$ and let v be any uncolored vertex. If v has four colored neighbors, then there exists a partial coloring π_1 of G using colors in $[5]$ and a color $\alpha \in [5]$ so that π_1 has the same domain as π, $\pi(x) \neq \pi_1(x)$ implies $x \in N(v)$ or $x \in N(N(v))$, and α is feasible for v under π_1.*

Moreover, in all the above lemmata, both π_1 and α can be found in $O(1)$ time.

2 A Sketch of the Proofs of Lemmata 1.2, 1.3, and 1.4

Lemmata 1.2,1.3: Notice that when extending a partial coloring to a vertex v, if v has less than 3 colored neighbors, then there always exists a feasible color out of the five colors. In the case that v has three colored neighbors, then we may not have a feasible color when two of these neighbors have the same color. In this case, we recolor a neighbor of v that is a single vertex, if such a vertex exists. Otherwise, it can be shown that there always exists a feasible color for v.

Lemma 1.4: In the case that all the four neighbors of v are colored, it is more difficult to see which vertices have to be recolored so that a feasible color for v can be found. In this direction, we investigate the colors in the 2-neighborhood of v. Suppose that three neighbors of v have the same color but the other neighbor is different from these two. Assume without loss of generality that $\pi(w) = \pi(x) = \pi(y) = 1$, and $\pi(z) = 2$. Considering the number of possible dangerous $1, \beta$ dangerous cycles, there may be no feasible color for v. We consider two cases. When any of $\{w, x, y\}$ is a single vertex, we recolor a single vertex from one of w, x, y. Depending on the new color of say, w.l.o.g, w, one can find a color that is feasible for v. When none of $\{w, x, y\}$ are single vertices, then three $1, \beta$ dangerous cycles must exist, for otherwise there is a feasible color for v. This implies that $\{w, x, y\}$ contain two differently colored neighbors. Assume, w.l.o.g, that w has neighbors colored so that color 4 appears at negihbors w_1 and w_2 and that color 5 appears at neighbor w_3. Then, we can recolor w if color 2 or 3 is missing in the neighbourhood of w_1 or w_2. Otherwise, it can be noticed that one of w_1 or w_2 should be a single vertex. This allows us to recolor w_1 from 4 to 5. This helps us in similarly exploring the colors in the neighborhood of w_1 and w_3. It then holds that either w_3 is also single or there exists a feasible color for v. The full report, dealing with all possible cases, is available as [10].

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