A tight lower bound for convexly independent subsets of the Minkowski sums of planar point sets*

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Recently, Eisenbrand, Pach, Rothvoß, and Sopher [1] studied the function $M(m,n)$, which is the largest cardinality of a convexly independent subset of the Minkowski sum of some planar point sets P and Q with $|P| = m$ and $|Q| = n$. They

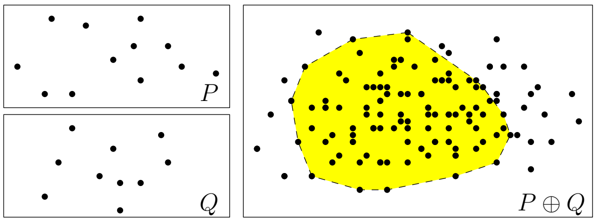

Figure 1: An example.

proved that $M(m,n) = O(m^{2/3}n^{2/3} + m + n)$, and asked whether a superlinear lower bound exists for $M(n,n)$. The quantity $M(n,n)$ gives an upper bound for the largest convexly independent subset of $P \oplus P$, and it is related to the convex dimension of graphs, proposed by Halman, Onn, and Rothblum [3]. Figure 1 shows an example. In this note, we show that the upper bound presented in [1] is the best possible apart from constant factors.

Theorem 1. *For every $m, n \in \mathbb{N}$, there exist point sets $P, Q \subset \mathbb{R}^2$ with $|P| = m$, $|Q| = n$ such that the Minkowski sum $P \oplus Q$ contains a convexly independent subset of size $\Omega(m^{2/3}n^{2/3} + m + n)$.*

Definitions. The *Minkowski sum* of two sets $P, Q \subseteq \mathbb{R}^d$ is defined as $P \oplus Q = \{p + q \mid p \in P, q \in Q\}$. A point set $P \subseteq \mathbb{R}^d$ is *convexly independent* if every point in P is an extreme point of the convex hull of P.

Basic idea. Let n and m be integers. Let P be a planar point set that maximizes the number of point-line incidences between m points and n lines. Erdős [2] showed that for $m, n \in \mathbb{N}$, there exist a set P of m point and a set L of n lines in the plane with $\Omega(m^{2/3}n^{2/3} + m + n)$ point-line incidences. A *point-line incidence* is a pair of a point p and a line ℓ such that $p \in \ell$ (that is, p lies on ℓ). Szemerédi and Trotter [5] proved that this bound is the best possible, confirming Erdős' conjecture (see [4] for the currently known best constant coefficients).

Sort the lines in L by the increasing order of their slopes (break ties arbitrarily). Denote by P_i the set of points in P that are incident to the ith line in L. Consider a polygonal chain C consisting of $|L|$ line segments such that the ith segment s_i has the same slope as the ith line of L. Since we sorted the lines in L by their slopes, C is a (weakly) convex chain. Set the length of each line segment to be at least the diameter of the point set P. The chain C has $n+1$ vertices including two endpoints. Now we can describe our point set $Q = \{q_1, \ldots, q_n\}$. The ith point q_i is placed on the plane so that the points in $P_i \oplus \{q_i\}$ all lie on s_i. This concludes the construction of Q. See Figure 2 for an illustration.

The number of points in $P \oplus Q$ that lie on C is $\Omega(m^{2/3}n^{2/3} + m + n)$ since if $p \in P_i$ then $p + q_i \in s_i \subseteq C$. Thus in the above construction, $(P \oplus Q) \cap C$ is a subset of $P \oplus Q$, it contains $\Omega(m^{2/3}n^{2/3} + m + n)$ points in (weakly) convex position.

Fine tuning. The point set $(P \oplus Q) \cap C$ is not necessarily convexly independent for two reasons:

1. Some of the lines in L may be parallel.
2. For each i, the points in $(P \oplus Q) \cap s_i$ are collinear.

We next describe how to overcome these issues.

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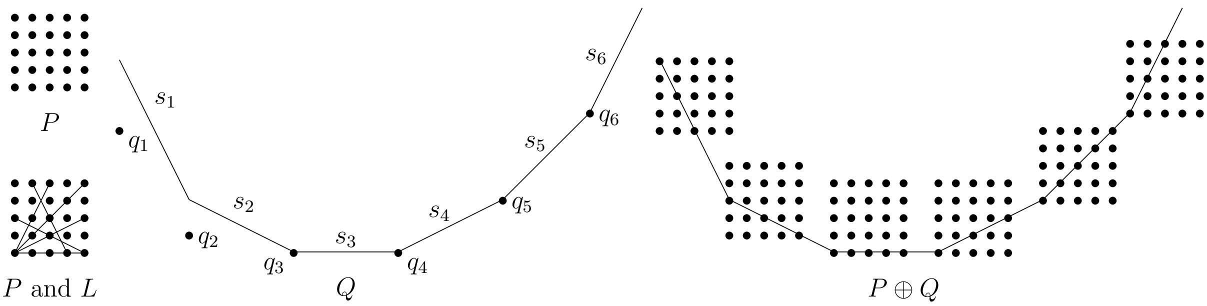

Figure 2: Basic idea for our construction.

For the first issue, we apply a projective transformation to P and L. A generic projective transformation maps P to a set of real points, and L to a set of pairwise nonparallel lines. Since projective transformations preserve incidences, the number of incidences remains $\Omega(m^{2/3}n^{2/3} + m + n)$. By applying a rotation, if necessary, we may assume that no line in L is vertical. Therefore, without loss of generality we may assume that all lines of L have different non-infinite slopes. As before we sort the lines in L in the increasing order by their slopes.

For the second issue, we apply the following transform to P and L (after the projective transformation and the rotation above): Each point (x, y) in the plane is mapped to $(x, y + \varepsilon x^2)$ for a sufficiently small positive real number ε. Then the ith line $y = a_i x + b_i$ is mapped to the convex parabola $y = \varepsilon x^2 + a_i x + b_i$. By scaling the whole configuration, we may assume that the x-coordinates of all points of P are between 0 and 1. Then, the gradient of the ith parabola is a_i at $x = 0$ and $a_i + 2\varepsilon$ at $x = 1$. Let ε be so small that the intervals $[a_i, a_i + 2\varepsilon]$ are all disjoint: Namely, the gradient of the ith parabola at $x = 1$ is smaller than the gradient of the $(i + 1)$st parabola at $x = 0$ (or more specifically it is enough to choose $\varepsilon = \min\{(a_i - a_{i-1})/3 \mid i = 2, \dots, n\}$). Therefore, instead of constructing a convex chain by line segments, we construct a convex chain C consisting of convex parabolic segments: The ith segment is a part of an expanded copy of the ith parabola (containing the piece between $x = 0$ and $x = 1$). From the discussion above, these parabolic segments together form a strictly convex chain and we can construct the point set Q in the same way as the previous case. Thus, for these P and Q, the set $(P \oplus Q) \cap C$ is a convexly independent subset in $P \oplus Q$ of size $\Omega(m^{3/2}n^{3/2} + m + n)$. Q.E.D.

An open problem. Let $M_k(n)$ denote the maximum convexly independent subset of the Minkowski sum $\bigoplus_{i=1}^{k} P_i$ of k sets $P_1, P_2, \dots, P_k \subset \mathbb{R}^2$, each of size n. Our lower bound in the case $m = n$, combined with the upper bound in [1] shows that $M_2(n) = \Theta(n^{4/3})$. Determine $M_k(n)$ for $k \geq 3$.

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