Advances in the Erdős-Sós conjecture for spiders

C. Balbuena[*], M. Guevara[*], J.R. Portillo[†‡] and P. Reyes[†]

The Erdős-Sós conjecture [3] says that a graph G with n vertices and $e(G) > n(k-1)/2$ edges contains all trees of size k.

Obviously, it holds for stars of size k, since if $e(G) > n(k-1)/2$, then some vertex in G must have degree at least k. Further, as Woźniak mentioned in [9], the above conjecture holds for the following trees: paths [4], double stars, comets (trees obtained from a star and a path by identifying one leaf of the star with one leaf of the path), caterpillars, and trees with a vertex adjacent to t leaves, where $t \geq (k-1)/2$. In [5], it is proved that the Erdős-Sós conjecture holds for some kind of spiders (trees with an unique vertex of degree greater than two), namely spiders with three legs or spiders with legs of length at most four.

The conjecture has been also proved for graphs G of size $e(G) > n(k-1)/2$, verifying one of the following assertions: the girth of G is at least 5 [2], the graph G does not contain the cycle C_4 [8], the graph G does not contain $K_{2,\lfloor k/18 \rfloor}$ as a subgraph [6] or the girth of the complement graph of G is at least 5 [7]. Recently, Balbuena *et al.* [1] have proved that any connected graph G of minimum degree $\delta(G) \geq k-1$ and maximum degree $\Delta(G) \geq k$ contains every tree of size k.

Let $S_{\ell_1,\ell_2,\ldots,\ell_f}$ be a *k-spider* of f legs $P_1, P_2, \ldots, P_f$ such that $e(P_i) = \ell_i$, and $\ell_1 + \ell_2 + \ldots + \ell_f = k$. k is the size of the k-spider. We may assume that $\ell_1 \leq \ell_2 \leq \ldots \leq \ell_f$. Let G be a graph with $e(G) > |V(G)|(k-1)/2$. Let H be a minimal induced subgraph of G such that $e(H) > |V(H)|(k-1)/2$. By the minimality, H is connected and $d_H(v) \geq k/2$ for every $v \in V(H)$ i.e., $\delta(G) \geq k/2$. If H has a copy of $S_{\ell_1,\ell_2,\ldots,\ell_f}$, so does G. Now, we state our first main result:

Theorem 1 *If H is hamiltonian, then H contains any k-spider $S_{\ell_1,\ell_2,\ldots,\ell_f}$*

By $g(n,k)$ Erdős and Gallai [4] denoted the maximum number of edges of a graph G on n vertices containing no cycles with more than k edges. Moreover, these authors proved the following result.

Theorem 2.7

$$g(n,k) \leq \frac{1}{2}(n-1)k, \text{ for } 2 \leq k \leq n.$$

Thus if $e(G) > (n-1)k/2$ then G contains a cycle with at least $k+1$ edges.

[*]Departament de Matemàtica Aplicada III. Universitat Politècnica de Catalunya, Spain
[†]Departamento de Matemática Aplicada I. Universidad de Sevilla, Spain
[‡]josera@us.es

Fan and Sun [5] used Theorem 2.7 to note that every graph G with $e(G) > n(k-1)/2$ has a circumference of length at least k. This is clear because $e(G) > n(k-1)/2 > (n-1)(k-1)/2$. Then they used this observation to prove that every graph with $e(G) > n(k-1)/2$ contains any k-spider of three legs. We think that similar arguments can be used to prove that a graph G with $e(G) > n(k-1)/2$ contains any k-spider of four legs. In fact, we have proved that every graph G with $e(G) > 9n/2$ contains the spider $S_{1,2,2,5}$. From here, it is easy to obtain the next result.

Theorem 2 *Let G be a graph with $e(G) > |V(G)|(k-1)/2$. If $k \leq 10$, then G contains any k-spider of four legs.*

As a first step to generalize the previous theorem to any k, we have proved that the Erdős-Sós conjecture is valid for all the spiders with four legs, if one of the legs has length one.

Theorem 3 *If G is a graph with $e(G) > |V(G)|(k-1)/2$, then G contains any k-spider $S_{1,\ell_2,\ell_3,\ell_4}$*

The proof of the last theorem contains some interesting techniques, wich can be used in some cases to prove the Erdős-Sós conjecture for spiders of four legs.

References

[1] C. Balbuena, A. Márquez, and J. R. Portillo, *A sufficient degree condition for a graph to contain every tree of size k, (submitted).*

[2] S. Brandt and E. Dobson, *The Erdős-Sós conjecture for graphs of girth 5*, Discrete Math. 150 (1996) 411–414.

[3] P. Erdős, *Extremal problems in graph theory*, in: M. Fiedler (Ed.), Theory of Graphs and its Applications, Academic Press, 1965, pp. 2936.

[4] P. Erdős and T. Gallai, *On maximal paths and circuits of graphs*, Acta Math. Acad. Sci. Hungar. 10 (1959) 337–356.

[5] G. Fan and L. Sun, *The Erdős-Sós conjecture for spiders*, Discr. Mathematics 307 (2007) 3055-3062.

[6] P.E. Haxell, *Tree embeddings*, J. of Graph Theory 36 (2001) 121–130.

[7] G. Li, A. Liu, and M. Wang, *A result of Erdős-Sós conjecture*, Ars Combin. 55 (2000) 123–127.

[8] J.F. Saclé and M. Woźniak, *The Erdős-Sós conjecture for graphs without* C_4, J. of Combin. Theory Ser. B 70, (1997) 367–372.

[9] M. Woźniak, *On the Erdős-Sós conjecture*, J. of Graph Theory, 21(2), (1996) 229–234.