On an empty triangle with the maximum area in planar point sets

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1 Introduction

We deal with only finite point sets \( P \) in the plane in general position. A point set is convex or in convex position if it determines a convex polygon. A convex subset \( Q \) of \( P \) is said to be empty if no point of \( P \) lies inside the convex hull of \( Q \). An empty convex subset of \( P \) with \( k \) elements is also called a \( k \)-hole of \( P \).

Let \( P \) be an \( n \) planar point set in general position. For a subset \( Q \) of \( P \), denote the area of the convex hull of \( Q \) by \( A(Q) \). In [3], we considered the ratio between the maximum area of 3-holes (empty triangles) \( T \) of \( P \) and the whole area \( A(P) \). Namely, let

\[
F(P) = \max_{T \subset P} \frac{A(T)}{A(P)}
\]

and define \( f(n) \) as the minimum value of \( F(P) \) over all sets \( P \) with \( n \) points.

Then we obtained the following result where \( c \) is some constant:

**Theorem A.**

\[
\frac{23}{(37 + 3\sqrt{5})n + c} \leq f(n) \leq \frac{1}{n - 1} \quad \text{for any } n \geq 25.
\]

In this talk, we improve on the lower bound of \( f(n) \). To achieve the aim we consider the existence of 5-holes of point sets in the next section.

2 The existence of 5-holes

Let \( V(P) \) be a set of vertices; a subset of \( P \) on the boundary of the convex hull of \( P \). Then we obtain the next proposition.

**Proposition 1.** If \( |V(P)| \geq 5 \) for any 7 or 8 point set \( P \), we have a 5-hole of \( P \).

It is well-known that Harborth [2] proves that any 10 point set contains a 5-hole and the bound is tight. The next result by using Proposition 1 shows a sufficient condition for the existence of a 5-hole of a 9 point set. An ear of \( P \) is three consecutive vertices on the convex hull boundary of \( P \).

**Proposition 2.** Any 9 point set \( P \) contains a 5-hole if it has an empty ear.
Figure 1 gives an 8 point set in general position with an empty ear, containing no 5-hole. And we obtain the following lemma by using Proposition 2.

**Lemma 1.** Any 17 point set \( P \) contains two 5-holes with disjoint interiors or one 6-hole.

We give a 12 point set in general position in Fig. 2 which contains neither two 5-holes with disjoint interiors nor one 6-hole.

### 3 Result

For \( n \) point sets \( P \) in convex position, the value \( f^{\text{conv}}(n) \) is defined in a similar way to \( f(n) \). The following lemma is proved in [1].

**Lemma A.** For point sets in convex position with 5 elements and with 6 elements,

\[
  f^{\text{conv}}(5) = \frac{1}{\sqrt{5}} \quad \text{and} \quad f^{\text{conv}}(6) = \frac{4}{9}.
\]

From the Polar Partition of \( P \) with the apex a vertex of \( P \) we have \( \left\lfloor \frac{n-2}{15} \right\rfloor \) convex cones with disjoint interiors, each of which contains exactly 17 points of \( P \). By Lemma 1 each such convex cone contains two 5-holes with disjoint interiors or one 6-hole. We improve on the lower bound by virtue of Lemma A:

**Theorem 1.**

\[
  f(n) \geq \frac{1}{2n-5-(6-2\sqrt{5})\left\lfloor \frac{n-2}{15} \right\rfloor} \geq \frac{15}{(24+2\sqrt{5})n+c} \quad \text{for any } n \geq 17.
\]

![Fig. 1.](image1.png)  
![Fig. 2.](image2.png)

### References

