Upward Topological Book Embeddings of DAGs

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Abstract

Let G be a directed acyclic graph. An upward (k, h)-topological book embedding of G is an upward book embedding on k pages of a subdivision of G where every edge is replaced by a path having at most $h+2$ vertices. In this extended abstract it is shown that every DAG with n vertices admits an upward $(d+1, 2\lceil \log_d n \rceil - 1)$-topological book embedding, where d is any integer such that $d \geq 2$. The result extends to the upward case well-known theorems for topological book embeddings of undirected graphs [6, 9].

1 Introduction

Let k be a positive integer. A *k-page book embedding* of a graph is a total ordering of its vertices and a partition of its edges into k sets, called *pages*, such that no two edges in the same page cross. Two edges cross if their endvertices alternate with each other in the total ordering. A k-page book embedding of a directed acyclic graph (DAG) is *upward* if for every oriented edge (u, v) vertex u precedes v in the total ordering of the vertices. The study of (upward) book embeddings finds applications in VLSI design, fault-tolerant processing, parallel process scheduling, and others. See, e.g., [4] for a survey on book embeddings.

A *h-subdivision* of a DAG G is obtained by replacing each directed edge (u, v) of G with a directed path from u to v having at most $h+2$ vertices; the vertices of the path distinct from u and v are called *division vertices*. An *upward (k, h)-topological book embedding* of G is an upward k-page book embedding of a h-subdivision of G; note that an upward $(k, 0)$-topological book embedding of G is an upward k-page book embedding of G.

We investigate trade-offs between number of pages and number of division vertices per edge in upward topological book embeddings of DAGs. In [8] it has been proved that every upward planar digraph has an upward $(2, 1)$-topological book embedding (note that not all planar graphs have a $(2, 0)$-topological book embedding [1]). This paper proves the following.

Theorem 1 *Let d be any integer such that $d > 1$. Every DAG with n vertices has an upward $(d+1, 2\lceil \log_d n \rceil - 1)$-topological book embedding.*

It may be worth recalling that the undirected version of our problem has been quite extensively studied. A (k, h)-topological book embedding of an undirected graph G is a k-page book embedding of a subdivision of G where every edge is split with at most h division vertices. In [2] it was proved that every undirected planar graph admits a $(2, 1)$-topological book embedding. Enomoto and Miyauchi [6] show that every undirected graph G with n vertices and m edges admits a $(3, 2\lceil \log_2 n \rceil + 1)$-topological book embedding; it is also proved in [7] that a total number of $\Omega(m \log n)$ division vertices along the edges of G may be needed in some cases. Miyauchi [9] extends the result of [6] by proving that G has a $(d+1, \lceil \log_d n \rceil)$-topological book embedding, for every integer $d > 1$. Dujmović and Wood [5] improve this last result by proving that G has a $(d+1, h)$-topological book embedding with $h = O(\log_d \min\{sn(G), qn(G)\})$, where $sn(G)$ and $qn(G)$ are the stack number and the queue number of G, respectively. The result of Theorem 1 is the directed counterpart of the result of Miyauchi [9].

2 Sketch of Proof of Theorem 1

A sketch of proof of Theorem 1 is given below. A detailed proof can be found in [3]. Let G be a digraph with n vertices and let $d > 1$ be an integer. Assume first that $n = d^h$ for some positive integer h.

The first step of the proof constructs a special type of directed acyclic graph called *foliage graph*. Let $T_{d,h}$ be a complete d-ary tree of height h. Roughly, the foliage graph $F(T_{d,h})$ is defined as follows. Perform a Eulerian tour traversal of $T_{d,h}$ starting from its root; each time a vertex μ of $T_{d,h}$ is encountered in the tour, a new vertex is added to $F(T_{d,h})$. The vertices of $F(T_{d,h})$ are sorted according to the order they are added to $F(T_{d,h})$. Such an ordering is denoted as

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σ_F. Every vertex v of the foliage graph is therefore associated with a vertex μ_v in $T_{d,h}$. Two vertices u and v of the foliage graph are connected by an edge oriented from u to v if and only if: (i) the vertices μ_u and μ_v are adjacent in $T_{d,h}$ and (ii) u precedes v in σ_F. See Figure 1(b) for an example. Note that, while an internal vertex of $T_{d,h}$ corresponds to $d+1$ vertices of $F(T_{d,h})$, each leaf of $T_{d,h}$ corresponds to exactly one vertex of $F(T_{d,h})$. A *terminal vertex* of $F(T_{d,h})$ is a vertex that corresponds to a leaf of $T_{d,h}$.

The second step maps the vertices of G to the terminal vertices of $F(T_{d,h})$. We compute a topological sorting of the vertices of G. Each vertex v of G is mapped to a distinct terminal vertex $\phi(v)$ of $F(T_{d,h})$. The mapping is such that if u precedes v in the topological sorting of G, then $\phi(u)$ precedes $\phi(v)$ in σ_F. See Figure 1 for an example.

The third step maps the edges of G to paths of $F(T_{d,h})$. These paths are chosen in such a way that every edge $e = (u, v)$ of G is mapped to a path $\phi(e)$ oriented from $\phi(u)$ to $\phi(v)$ in $F(T_{d,h})$. Path $\phi(e)$ is called the *foliage path* of e. See Figure 1 for an example. The mapping of the edges of G to the foliage paths is used to define a subdivision G' of G: Each edge $e = (u, v)$ of G is replaced with a directed path $\pi(e)$ from u to v of length equal to the length of $\phi(e)$. By exploiting the completeness of $T_{d,h}$, it is possible to prove that one can choose foliage paths having at most $2h-1$ internal vertices. Therefore, G' is a $(2\log_d n - 1)$-subdivision of G.

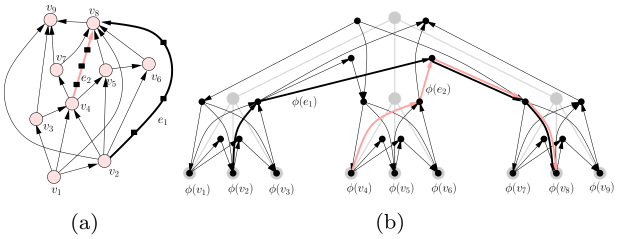

Figure 1: (a) A DAG G. The indices of the vertices define a topological sorting. Edges e_1 and e_2 are highlighted. The little squares represents division vertices induced by the mapping $\phi(e_1)$ and $\phi(e_2)$. (b) A foliage graph $F(T_{3,2})$. The light grey vertices/edges show $T_{3,2}$, while black edges are the edges of $T_{3,2}$. The mapping of the vertices of G is shown. Paths $\phi(e_1)$ and $\phi(e_2)$ are highlighted.

The fourth step computes an upward book embedding γ of $F(T_{d,h})$ on $d+1$ pages where the vertices along the spine are ordered according to σ_F (see [3] for details). Consider now an edge $e = (u, v)$ of G. The embedding of path $\phi(e)$ on at most $d+1$ pages in γ naturally defines a corresponding embedding on $d+1$ pages of $\pi(e)$. Since $\phi(e)$ is upward, also $\pi(e)$ is upward. In order to completely define an upward book embedding of G' and therefore an upward topological book embedding of G we have to define an ordering for the division vertices corresponding to the same non-terminal vertex of the foliage graph. Namely, a same edge of the foliage graph may be part of different foliage paths of different edges of G. As described in [3], a suitable ordering of the division vertices can be computed such that no two edges of G' in the same page cross. It follows that G has an upward $(d+1, 2\log_d n - 1)$-topological book embedding.

It remains to consider the case that there is no integer h such that $n = d^h$. In this case we augment G to a DAG G^* with $d^{\lceil \log_d n \rceil}$ vertices and compute an upward $(d+1, 2\lceil \log_d n \rceil - 1)$-topological book embedding of G^* by the technique described above. Removing the vertices and edges added to G^*, we obtain an upward $(d+1, 2\lceil \log_d n \rceil - 1)$-topological book embedding of G. This completes the proof.

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