Arrangements of n points whose incident-line-numbers are at most $n/2$

Jin Akiyama
Research Institute of Educational Development
Tokai University
fwjb5117@mb.infoweb.ne.jp

Hiro Ito
Department of Communications and Computer Engineering
Graduate School of Informatics
Kyoto University
itohiro@kuis.kyoto-u.ac.jp

Midori Kobayashi
University of Shizuoka
midori@u-shizuoka-ken.ac.jp

Gisaku Nakamura
Research Institute of Educational Development
Tokai University

We consider a set X of n points in the plane, not all in a line, and the set of lines $\mathcal{L}$ spanned by X, where we say that a line is spanned by X if it passes through at least two elements of X.

For a point P of X, we denote by $t(P)$ the number of lines in $\mathcal{L}$ which are incident to P, and call it the incident-line-number of P. We define $t(X)$ to be $\mathrm{Max}_{P \in X} t(P)$, and call it the maximum incident-line-number of X.

What values does $t(X)$ take for various arrangements X of n points? If X is an n point set, any three of them are not on a line, then we have $t(X) = n - 1$. So the maximum value for $t(X)$ is $n - 1$.

What is the minimum value for $t(X)$? Dirac showed the following inequality for $t(X)$ and posed a conjecture.

Theorem A ([1], p. 226) For any set X of n points, not all on a line, we have $t(X) \geq \sqrt{n}$.

Conjecture B ([1], p. 226)[1] For any set X of n points, not all on a line, we have $t(X) \geq \lceil n/2 \rceil$.

Some counterexamples were shown for a few small values of n $(9, 15, 19, 25, 31, 37)$ by Grünbaum ([3], p. 25) and an infinite family of counterexamples was constructed by Felsner ([5], p. 313): a set X of $n = 6k + 7$ $(k \geq 2)$ points with $t(X) \leq \lfloor n/2 \rfloor - 1$ when k is even, and with $t(X) \leq \lfloor n/2 \rfloor$ when k is odd.

In this talk, we consider the following problem which will show that Conjecture B does not hold for all odd numbers $n \geq 8$.

[1] In Dirac's paper [1] (p. 226), he uses the notation $[\quad]$ instead of $\lceil\quad\rceil$.

Problem For every integer $n \geq 8$, construct a set X of noncolinear n points satisfying $t(X) \leq \lfloor n/2 \rfloor$.

To solve the problem, we consider the following cases: (i) n is even, ≥ 8, (ii) $n = 4k+9$, $k \geq 0$, (iii) $n = 12k+15$, $k \geq 0$, (iv) $n = 12k+19$, $k \geq 0$, (v) $n = 12k+23$, $k \geq 0$, (vi) $n = 11$.

We construct solutions for (i) - (iii); (iv) is the Felsner's arrangement; (vi) is a special construction using a BIBD which is shown in another talk; and (v) remains open.

Figure 1 and 2 are examples of sets for (ii) and (iii), respectively.

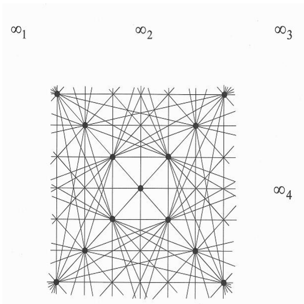

Figure 1. $n = 17, t(X) = 8$.

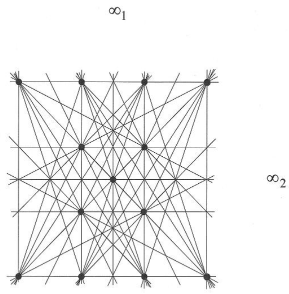

Figure 2. $n = 15, t(X) = 7$.

The following conjecture and problem are proposed but still unsolved.

Conjecture C (strong Dirac conjecture) ([5], p. 313) There is a constant c such that any set X of n points, not all on a line, has an element incident to at least $n/2 - c$ lines spanned by X.

Problem D ([4], p. 29) Does any set X of n points, not all on a line, have an element incident to at least $n/3$ lines spanned by X?

References

[1] G. A. Dirac, Collinearity Properties of Sets of Points, Quart. J. Math. Oxford (2) 2 (1951), 221-227.

[2] B. Grünbaum, Arrangements of hyperplanes. Proc. Second Louisiana Conf. on Combinatorics, Graph Theory and Computing, R. C. Mullin et al., eds. Louisiana State University, Baton Rouge 1971, pp. 41 - 106.

[3] B. Grünbaum, Arrangements and Spreads. Conference Board of the Mathematical Sciences, Regional Conference Series in Mathematics, Number 10. Amer. Math. Soc., Providence, RI, 1972, 114 pp.

[4] V. Klee and S. Wagon, *Old and New Unsolved Problems in Plane Geometry and Number Theory*, The Mathematical Association of America, 1991.

[5] P. Brass, W. Moser, J. Pach, *Research Problems in Discrete Geometry*, Springer, 1st ed. 2005.