Geometric realization of a triangulation on the Klein bottle with one face removed

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A *map* is a fixed embedding of a graph on a surface F^2. A *triangulation* on a surface F^2 is a map on F^2 such that each face is bounded by a 3-cycle, where a k-cycle means a cycle of length k. We suppose that the graph of a map is always *simple*, i.e., with no multiple edges and no loops. Let M be a map on a surface F^2. A *geometric realization* of M is an embedding of F^2 into an Euclidian 3-space $\mathbb{R}^3$ with no self-intersection such that each face of M is a flat polygon.

Steinitz has proved that a spherical map has a geometric realization as a convex polygon if and only if its graph is 3-connected [6]. (The 3-connectedness is necessary to make a geometric realization as a convex polygon. However, in our definition of a geometric realization, two adjacent faces may lie on the same plane in $\mathbb{R}^3$.) Moreover, Archdeacon et al. have proved that every toroidal triangulation has a geometric realization [1]. In general, Grünbaum has conjectured that every triangulation on any orientable closed surface has a geometric realization [5], but Bokowski et al. have shown that a triangulation by the complete graph K_{12} with twelve vertices on the orientable closed surface of genus 6 has no geometric realization [3]. Hence Grünbaum's conjecture is no longer true now but it is still open for the orientable closed surface of genus from 2 to 5.

Let us consider nonorientable surfaces. It is known that nonorientable closed surfaces are not embeddable into $\mathbb{R}^3$. So, no map on nonorientable closed surfaces has a geometric realization. However, surfaces obtained from nonorientable closed surfaces by removing an open disk are embeddable into $\mathbb{R}^3$. (For example, the surface obtained from the projective plane by removing an open disk is isomorphic to the Möbius band.) Therefore, we can expect that a triangulation on such surfaces has a geometric realization. For simple notations, we call a triangulation on the projective plane a *projective triangulation*, one on the Klein bottle a *Klein triangulation* and one on the Möbius band a *Möbius triangulation*, respectively, throughout the paper. The following has been proved.

Theorem 1 (Bonnington and Nakamoto [2]) *Every projective triangulation G has a face f such that $G - f$ has a geometric realization.*

Let G be a projective triangulation. Although Theorem 1 says that G always has a face f which enables to $G - f$ has a geometric realization, we cannot choose any face of G as f since Brehm has shown a counterexample [4]. The left of Figure 1 shows a Möbius triangulation with no geometric realization obtained from a projective triangulation G by removing the face 123. Recently, we have characterized a face f of G such that $G - f$ has a geometric realization. Moreover, that characterizes Möbius triangulations which have a geometric realization.

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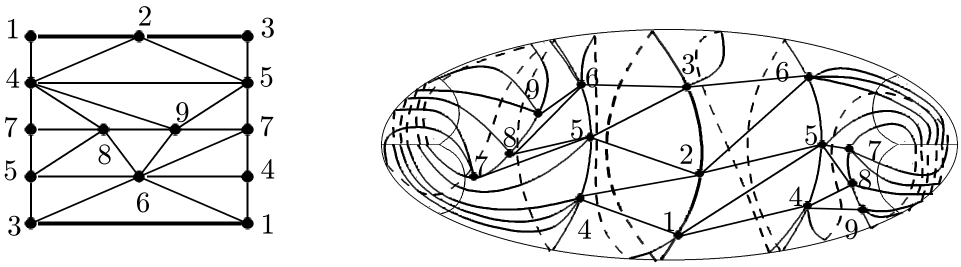

Figure 1: A Möbius triangulation M with no geometric realization and a Klein triangulation obtained from two Ms by pasting along their boundaries.

Now we consider the case of the Klein bottle. Let G be a triangulation on the nonorientable surface and let f be a face of G. If $G - f$ has a geometric realization, then f is said to be a *removable* face. Although every projective triangulation has a removable face by Theorem 1, there exists a Klein triangulation which has no removable face. We can construct such a triangulation from two Brehm's counterexamples by pasting along their boundaries since the Klein bottle is obtained from two Möbius bands by pasting along their boundaries. The right of Figure 1 shows a Klein triangulation which has no removable face.

We would like to characterize Klein triangulations which have a removable face. There exist 29 *irreducible* Klein triangulations which are minimal with respect to the number of edges [7]. For irreducible Klein triangulations, we have already proved the following.

Proposition 2 *Every irreducible Klein triangulation G has a face f such that $G - f$ has a geometric realization.*

In my talk, we would like to prove the following.

Theorem 3 *Every 5-connected Klein triangulation G has a face f such that $G - f$ has a geometric realization.*

Moreover, we conjecture that Theorem 3 can be improved with respect to the connectivity of triangulations.

References

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