Rainbow Colorings of Generalized Petersen Graphs

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Abstract

The *generalized Petersen graph* $GP(n,k)$ for positive integers n,k and $n \neq 2k$, $k \leq n-1$ is the graph with vertex set $V[GP(n,k)] = \{0,1,2,\ldots,(n-1),0',1',2',\ldots,(n-1)'\}$ and edge set $\{(i,i+1),(i,i'),\text{ and }(i',(i+k)')\}$. These edges are also called *rim*, *spoke*, and *hub* edges, respectively. Some examples are shown below. Note that by symmetry, $GP(n,k) = GP(n,n-k)$.

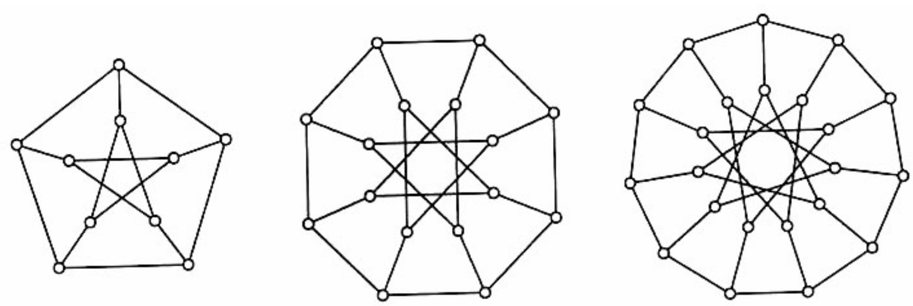

Figure 1. The graphs $GP(5,2), GP(8,3)$ and $GP(11,4)$

An *edged-coloring* of a graph G is a function c which assigns a color $c(x)$ to each edge $x \in E(G)$. An edge-coloring is called *proper* if $c(x) \neq c(y)$ for all pairs of incident edges $x,y \in E(G)$. In this paper, we define a proper edge-coloring of the generalized Petersen graph $GP(n,k)$ to be *rainbow* if for any chordless cycle of length $L \leq k+3$, each edge has a distinct color. Note that we consider only colorings which are proper.

Let $g(n,k)$ be the least integer such that there exists a rainbow coloring of $GP(n,k)$. Then clearly, $g(n,k)$ is at least $k+3$ for any n,k. In this paper, we prove that $g(n,k) = k+3$ when k divides n, when $k=2$, and when $k=1$ (for $n>3$). This value $g(n,k) \; = k+3$ is best possible, since any rainbow coloring of $GP(n,k)$ will necessarily have at least $k+3$ colors, to assure that in any cycles of length $L \leq k+3$, each edge has a distinct color.

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As an example, note that in the 5-coloring of $GP(5,2)$ shown in Figure 2, each edge of any 5-cycle has a distinct color. Therefore, $g(5,2) = 5$.

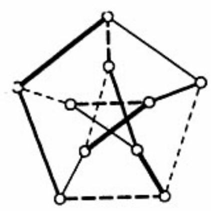

Figure 2. A rainbow coloring of $GP(5,2)$

In $P(n,k)$, we refer to the $(k+3)$-cycles comprised by k adjacent rim edges, two spoke edges and one hub edge as a *fundamental cycle*. In general, for $n \geq 3k$, $GP(n,k)$ contains no non-fundamental cycles, which leads us to the following:

Conjecture: $g(n,k) = k+3$ for $GP(n,k)$, $n \geq 3k$.

A similar problem has been studied by, Faudree, et.al. [FGS96]. In the paper a *rainbow coloring* is defined as a proper coloring of a graph such that in any C_4, chordless cycle of length four, each edge has a distinct color. The authors prove that for $n \geq 6$, the minimum number of colors needed for a rainbow coloring of $K_n \times K_n$ is n, denoted by $rb(n) = n$. In another paper [FGLS93] Faudree, et.al. considered the hypercube Q_n where $Q_k = Q_{k-1} \times K_2$ for $k \geq 1$ and $Q_0 = K_1$. They proved that for $n \geq 4$, $d \neq 5$, $rb(Q_d) = d$.

References

[FGLS93] Faudree, R.J., Gyáfás, A., Lesniak, L., Schelp, R.H.: Rainow coloring the cube. J. Graph Theory **17**, 607-612 (1993)

[FGS96] Faudree, R.J., Gyáfás, A., Schelp, R.H.: An edge coloring problem for graph products. J. Graph Theory **23**, 297-302 (1996)