Wide-diameter of a k-connected graph with diameter d

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Abstract

Let k be a positive integer and let G be a graph. For two distinct vertices $x, y \in V(G)$, the k-wide-distance $d_k(x, y)$ between x and y is the minimum integer l such that there exist k disjoint (x, y)-paths whose lengths are at most l. We define $d_k(x, x) = 0$. The k-wide-diameter $d_k(G)$ of G is the maximum value of the k-wide-distance between two vertices of G. In this paper we prove that for a fixed positive integers k and Δ, k-wide-diameter of a k-connected graph with diameter d and $\Delta(G) \leq \Delta$ is bounded by a polynomial of d of which degree is k (the coefficients of the polynomial are depend on k and Δ).

Introduction

In this note we consider a generalized distance arising from a system of k disjoint paths in k-connected graphs. We deal only finite undirected graphs possibly with multiple edges but without loops. For a graph G, let $V(G)$ and $E(G)$ be the vertex set of G and the edge set of G, respectively. Let $\Delta(G)$ denote the maximum degree of G. For $x, y \in V(G)$, the distance between x and y in G is denoted by $d(x, y)$. The diameter of G is denoted by $d(G)$.

Let k be a positive integer. Let G be a graph and let x, y be two distinct vertices of G. We denote the length of a path P by $l(P)$. A set of k internally disjoint (x, y)-paths is said to be an *Menger path system* of width k between x and y. Let $\mathcal{P}_k(x, y)$ be the collection of all Menger path systems of width k between x and y. We define the *k-wide-distance* between x and y by

$$d_k(x, y) = \min \left\{ \max_{1 \leq i \leq k} l(P_i) \,\middle|\, \{P_1, P_2, \ldots, P_k\} \in \mathcal{P}_k(x, y) \right\}$$

(if $\mathcal{P}_k(x, y) = \emptyset$, then we define $d_k(x, y) = \infty$) and define $d_k(x, x) = 0$. Moreover we define the *k-wide-diameter* of G by

$$d_k(G) = \max \left\{ d_k(x, y) | x, y \in V(G) \right\}.$$

Similarly, edge version Menger path system, edge version k-wide-distance $d_k^{\mathrm{e}}(x, y)$, and edge version k-wide-diameter $d_k^{\mathrm{e}}(G)$ are defined.

There are many results on (edge-)wide-diameter (see [2]).

Result

Let G be a k-(edge-)connected graph. By definition, both $d_k^{\mathrm{e}}(G)$ and $d_k(G)$ are not less than the diameter of G. One natural question is how big can $d_k^{\mathrm{e}}(G)$ and $d_k(G)$

be for a k-(edge-)connected graph G with diameter d. In the edge version case, we already have the following results.

Theorem A(Ando and Kaneko [1]). Let G be a 2-edge-connected graph with diameter d. Then

$$d_2^{\mathrm{e}}(G) \leq \begin{cases} \frac{1}{2}d^2 - d + 4 & (d \geq 4 \text{ even}) \\ \frac{1}{2}d^2 - d + \frac{9}{2} & (d \geq 3 \text{ odd}). \end{cases}$$

Moreover this bound is sharp.

Theorem B (Kojima, Ando and Kaneko [3]) Let k be a fixed positive integer. Let G be a k-edge-connected graph with diameter d. Then, there is a positive constant c_k such that

$$d_k^{\mathrm{e}}(G) \leq c_k d^k + O(d^{k-1}).$$

Theorem A shows that the constant $c_2 = \frac{1}{2}$. In the case $k \geq 3$, we know only $c_k \leq 2^{k-3}(k-1)!$. However the bound in Theorem B is sharp as the following sense. Remark (sharpness). Let k be a fixed positive integer. Then there exist a positive constant c'_k and a k-edge-connected graph G with diameter d such that

$$d_k^{\mathrm{e}}(G) \geq c'_k d^k + O(d^{k-1}).$$

However, in the vertex version case, there is no positive answer for the above question. In fact, we observe that $d_k(G)$ does not bounded by only d and k. For example, the graph $P_{n+1} + K_{k-1}$ has diameter 2 and k-wide-diameter n. (Its k-edge-wide-diameter is 3.) Hence we need to constrain graphs. We investigate graphs with bounded maximum degree and we get the following result.

Theorem 1 Let k and Δ be fixed positive integers. Let G be a k-connected graph with diameter d and $\Delta(G) \leq \Delta$. Then, there is a positive constant $c_{k,\Delta}$ such that

$$d_k(G) \leq c_{k,\Delta} d^k + O(d^{k-1}).$$

References

[1] K. Ando and A. Kaneko, 2-wide-diameter of 2-edge-connected graph with diameter d, preprint.

[2] R. J. Faudree, Some strong variations of connectivity, *Combinatorics, Paul Erdős is eighty, Vol.* 1, 125–144, *Bolyai Soc. Math. Stud., János Bolyai Math. Soc., Budapest,* 1993.

[3] T. Kojima, K. Ando and A. Kaneko, Edge-wide-diameter of graphs with diameter d, *Ann. Comb.* **6** (2002), no. 1, 57–64.