I216E: Computational Complexity and Discrete Mathematics

Answers and Comments on Report 1

HOANG, Duc Anh (1520016)

October 27, 2016

Ph.D Student @ Uehara Lab
Japan Advanced Institute of Science and Technology
hoanganhduc@jaist.ac.jp

Problem 1

For any given string x, we denote by $lo(x)$ and $oo(x)$ the indices of x in the pseudo-lexicographical ordering with length preferred and the usual lexicographical ordering, respectively. For example, we have $lo(\epsilon) = oo(\epsilon) = 1$, $lo(0) = oo(0) = 2$, $lo(1) = 3$, and $oo(00) = 3$. We also denote by $n < \infty$ when the number n is finite. Now, declare if each of the followings is true or false. If it is false, show a counterexample. In the followings, x denotes a string and n denotes a positive integer.

$$\forall x \exists n [\ |x| < \infty \to lo(x) < n\ ] \tag{1}$$

$$\exists n \forall x [\ |x| < \infty \to lo(x) < n\ ] \tag{2}$$

$$\forall x \exists n [\ |x| < \infty \to oo(x) < n\ ] \tag{3}$$

$$\exists n \forall x [\ |x| < \infty \to oo(x) < n\ ] \tag{4}$$

Problem 1 (Answer)

(1) $\forall x \exists n[\ |x| < \infty \to lo(x) < n\ ]$. True. Since x is of finite length, its index in the pseudo-lexicographical ordering $lo(x)$ is also finite, i.e., there exists some number n such that $lo(x) < n$.

(2) $\exists n \forall x[\ |x| < \infty \to lo(x) < n\ ]$. False. For a fixed number n, the string $x = 00\ldots0$ (containing $n+1$ 0s) has the index $lo(x) = 1 + 2 + 2^2 + \cdots + 2^n + 1 = 2^{n+1}$, which is much larger than n. The above formula for $lo(x)$ comes from the fact that there are 2^k binary strings of length k ($k \geq 0$).

(3) $\forall x \exists n[\ |x| < \infty \to oo(x) < n\ ]$. False. The statement does not hold for $x = 1$. In this case, the length of x is finite, but the index of x in the usual lexicographical ordering $oo(x)$ is infinite, i.e., there is no n such that $oo(x) < n$.

(4) $\exists n \forall x[\ |x| < \infty \to oo(x) < n\ ]$. False. For a fixed number n, one can always find a string x of finite length such that $oo(x) < n$ does not hold. For example, take $x = 1$.

Problem 2

The set $\mathbb{N}$ of natural numbers is enumerable. Now, prove that the set $2^{\mathbb{N}}$ of subsets of $\mathbb{N}$ is not enumerable by diagonalization. (Hint: For $S = \{1, 2, 3\}$ we have $2^S = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{2, 3\}, \{1, 3\}, \{1, 2, 3\}\}$.)

Problem 2 (Answer)

Suppose that the set $2^{\mathbb{N}}$ is enumerable. Hence, we can list elements of $2^{\mathbb{N}}$ as $N_0, N_1, N_2, \ldots$, where each N_i is a subset of $\mathbb{N}$ for some $i \in \mathbb{N}$. Next, we define the below table as follows: for $j \in \mathbb{N}$, put 1 to position (i, j) if $j \in N_i$; otherwise, put 0.

	0	1	2	$\ldots$	i	$\ldots$
N_0	1_0	0	1	$\ldots$	1	$\ldots$
N_1	0	1_0	0	$\ldots$	0	$\ldots$
N_2	0	1	0_1	$\ldots$	1	$\ldots$
$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$
N_i	1	0	1	$\ldots$	0_1	$\ldots$
$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$

Let $A = \{i \mid i \in \mathbb{N} \text{ and } i \notin N_i\}$. In the above table, 1 means $i \in A$ and 0 means $i \notin A$, where $i \in \mathbb{N}$.

Then, A is a subset of $\mathbb{N}$. It follows that $A = N_j$ for some $j \in \mathbb{N}$. But now, we have $j \in A$ if and only if $j \notin N_j = A$. Thus, the value at position (j, j) of the above table cannot be decided. Therefore, $2^{\mathbb{N}}$ is not enumerable.

Problem 3

In the slide of the second lecture, we prove the theorem that claims "The set R of real numbers is not countable." Now let replace every "real" by "rational". Then it seems that we prove the theorem that claims "The set R' of rational numbers is not countable." But, the set of all rational numbers is countable. Point out where is wrong.

Problem 3 (Answer)

JAPAN ADVANCED INSTITUTE OF SCIENCE AND TECHNOLOGY 1990

4. Undecidability and Diagonalization

4. 2. Diagonalization

Theorem:

The set R of real numbers is *not* countable.

[Proof by *diagonalization*]

Assume that P is countable; i.e., they are enumerated as $R = \{ R_0, R_1, R_2, R_3, \ldots \}$

Each R_i is in the form of $R_i = \ldots r_{i,4}' \, r_{i,3}' \, r_{i,2}' \, r_{i,1}' \, r_{i,0} \, . \, r_{i,1} \, r_{i,2} \, r_{i,3} \, r_{i,4} \ldots$ in decimal.

We define a number $X = 0.\, x_1 \, x_2 \, x_3 \ldots$ by

$$\begin{cases} x_i = 3 \text{ if } r_{i,i} = 1,2,4,5,6,7,8,9, \text{ or } 0 \\ x_i = 1 \text{ if } r_{i,i} = 3 \end{cases}$$

Then X is a real number, so it will appear as $X=R_i$ for some i.

But x_i is… 3? or 1?… we cannot decide it, which is a contradiction!

Therefore P is not countable!!

Ex.

$R_0 = 123.\underline{4}56\ldots$

$R_1 = 0.1\underline{3}1313\ldots$

$R_2 = 555.55\underline{5}5555\ldots$

$R_3 = 3.141\underline{5}92\ldots$

…

$X = 0.\,3133\ldots$

Figure 1: The original slide.

Problem 3 (Answer)

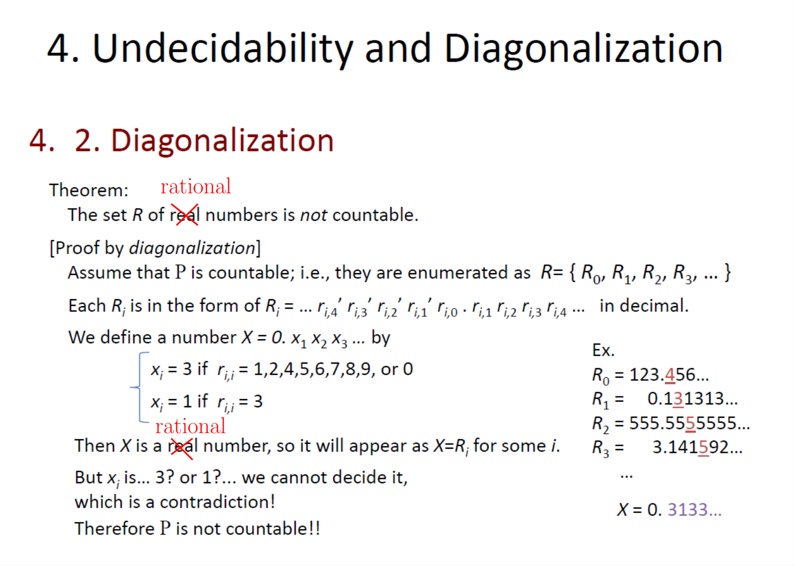

Figure 1: Replacing "real" by "rational".

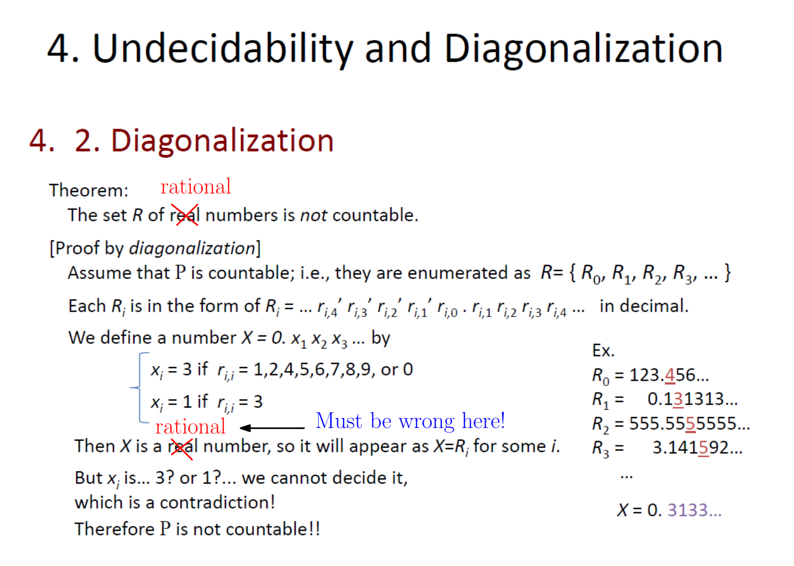

Figure 1: Replacing “real” by “rational”.

Problem 3 (Comments)

As we've seen, when we replace "real" by "rational", the proof becomes wrong. The constructed number X may not be a rational number, i.e., it can be irrational. Interestingly, this means that one can indeed construct an irrational number from an ordering of rational numbers.