Computational Origami

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Today…

1. Basic facts for unfolding
2. Polygons foldable two or more boxes
3. Common unfolding of regular polyhedra (or Platonic solids)
    1. Common unfolding of a regular tetrahedron and a cube
    2. Common unfolding of a regular tetrahedron and Johnson-Zalgaller solids

Common Unfolding of Regular Tetrahedron and Johonson-Zalgaller Solid

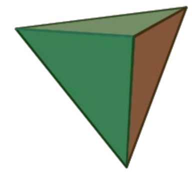
Reference:

- Yoshiaki Araki, Takashi Horiyama, and Ryuhei Uehara. Common Unfolding of Regular Tetrahedron and Johnson-Zalgaller Solid, Journal of Graph Algorithms and Applications, Vol.20, no.1, pp.101-114, February, 2016. DOI:10.7155/jgaa.00386

Common Unfolding

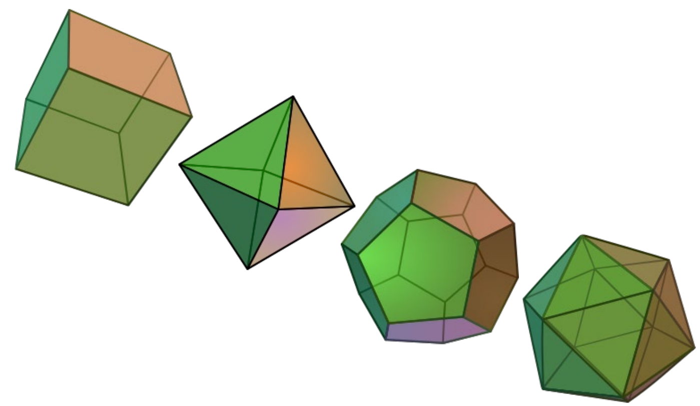
- Theorem： There exists **no** polygon **P** s.t.

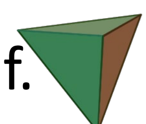
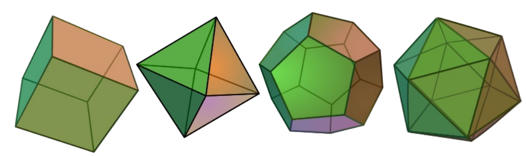
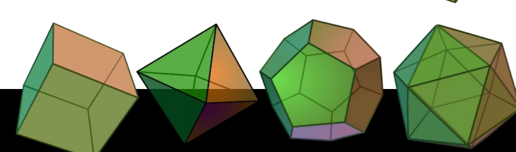
  (1) **P** is a **general** unfolding of
  a regular **tetrahedron**



[Horiyama, Uehara 2010]

  (2) **P** is an **edge** unfolding of
  - a regular **cube**
  - a regular **octahedron**
  - a regular **dodecahedron**
  - a regular **icosahedron**

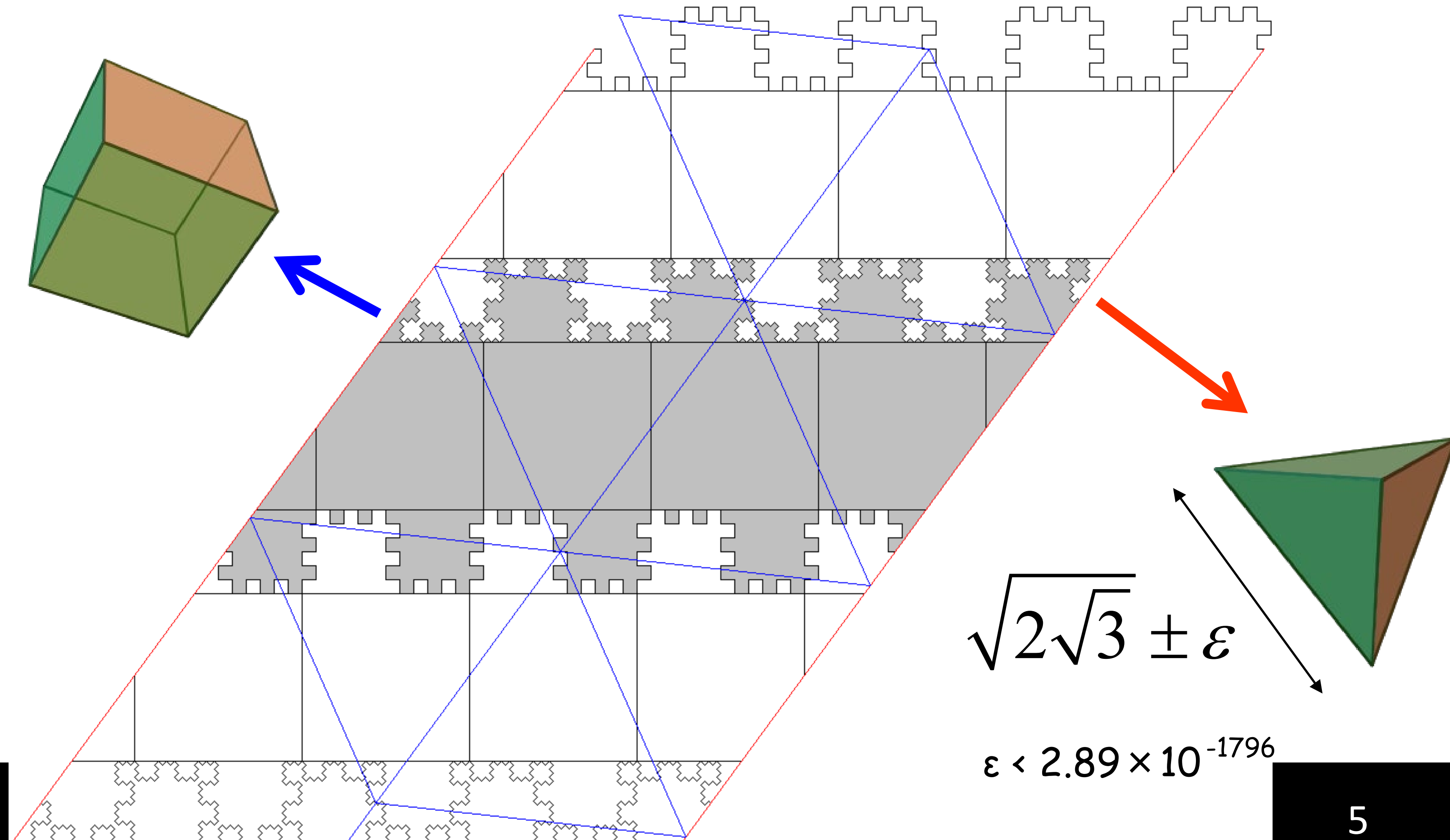


- Open
  - General unf.  v.s. General unf. 
  - General unf. among 

Common Unfolding

[Shirakawa, Horiyama, Uehara 2011]

- Cube v.s. Tetramonohedron

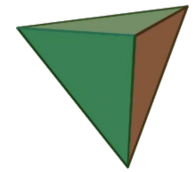


$$\sqrt{2\sqrt{3}} \pm \varepsilon$$

$\varepsilon < 2.89 \times 10^{-1796}$

Our work

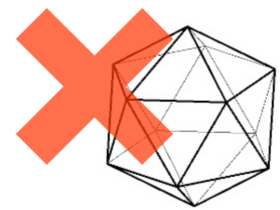
- Regular tetrahedron (general unfolding)



v.s. Convex polyhedron with **regular polygonal faces**

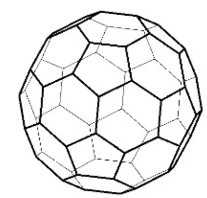
(edge unfolding)

- **Platonic solids**
  - face : 1 type of regular polygon
  - vertex : same face-arrangement

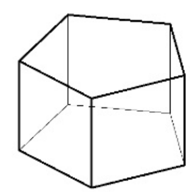


[Horiyama, Uehara 2010]

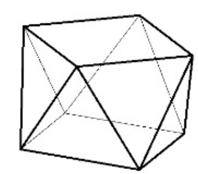
- **Archimedean solids**
  - face : 2+ types of regular polygons
  - vertex: same face-arrangement



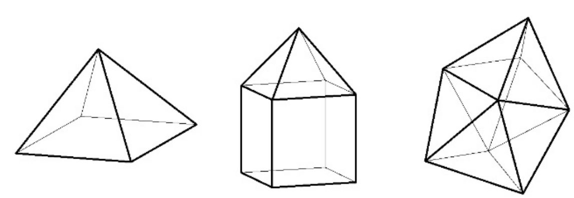
- ***n*-gonal prism**
  - face : 2 regular *n*-gons & *n* squares



- ***n*-gonal antiprism**
  - face : 2 regular *n*-gons & 2*n* regular triangles

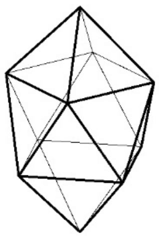


- **Johnson-Zalgaller solids**

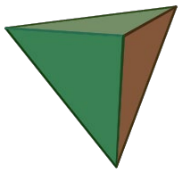


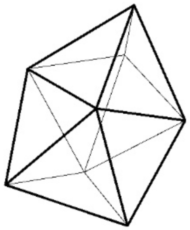
Our Results

- Johnson-Zalgaller solids : 92 solids

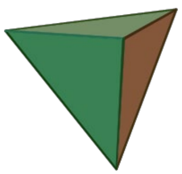


J17 (Gyroelongated square dipyramid)

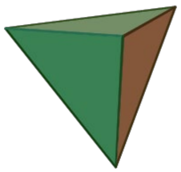
has **common unfolding** with 



J84 (Snub disphenoid)

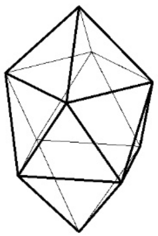
has **common unfolding** with 

- Archimedean solids
- *n*-gonal prism
- *n*-gonal antiprism
- other 90 Johnson-Zalgaller solids

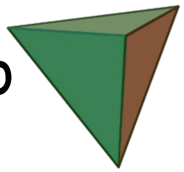
No common unfolding with 

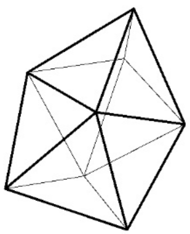
Our Results

- Johnson-Zalgaller solids : 92 solids



  - **J17** (Gyroelongated square dipyramid) 13,014 edge unfolding
    - 78 unf. : 1 way
    - 8 unf. : 2 ways
    - 1 unf. : **3 ways**

    } of folding into 



  - **J84** (Snub disphenoid) 1,109 edge unfolding
    - 32 unf. : 1 way
    - 5 unf. : **2 ways**

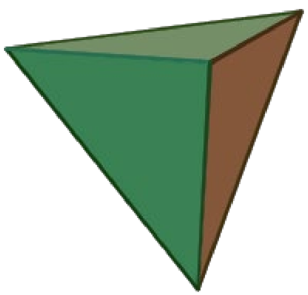
    } of folding into 

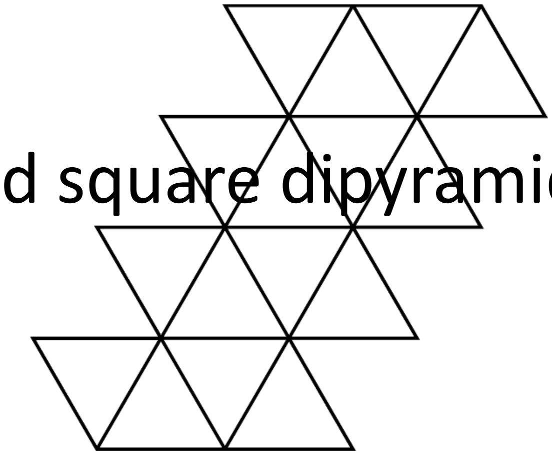
- Archimedean solids
- *n*-gonal prism
- *n*-gonal antiprism
- other 90 Johnson-Zalgaller solids

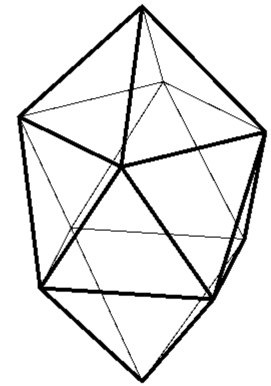
} **No** common unfolding with 

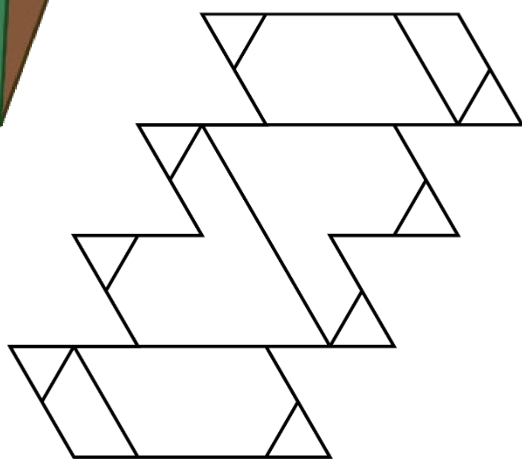
Our Results

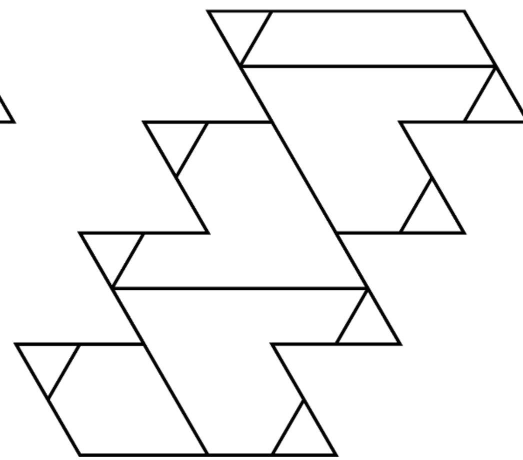
- J17 (Gyroelongated square dipyramid)

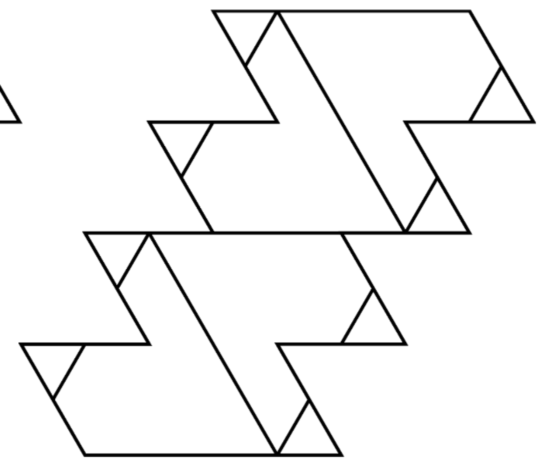






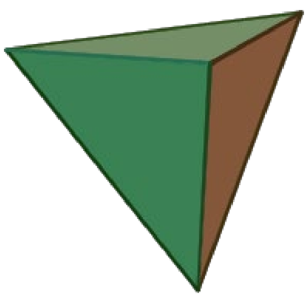


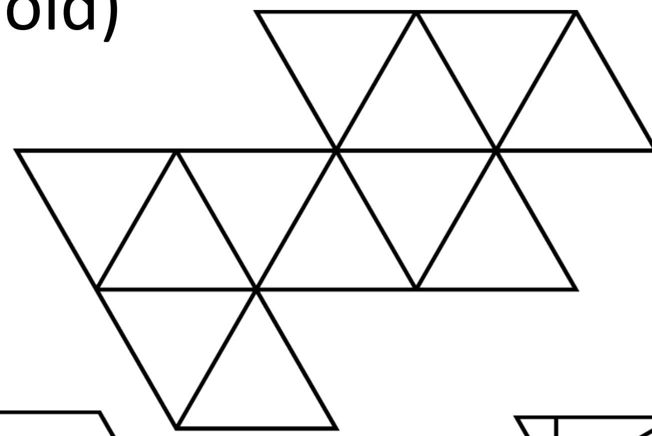


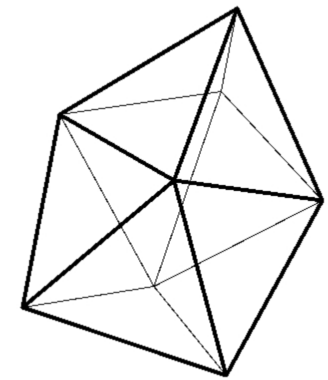


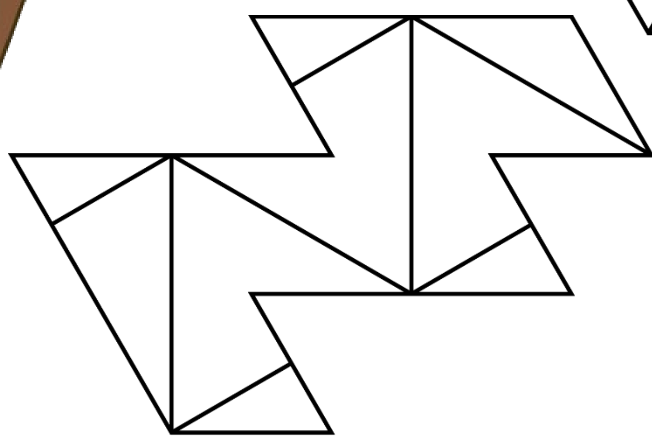
Our Results

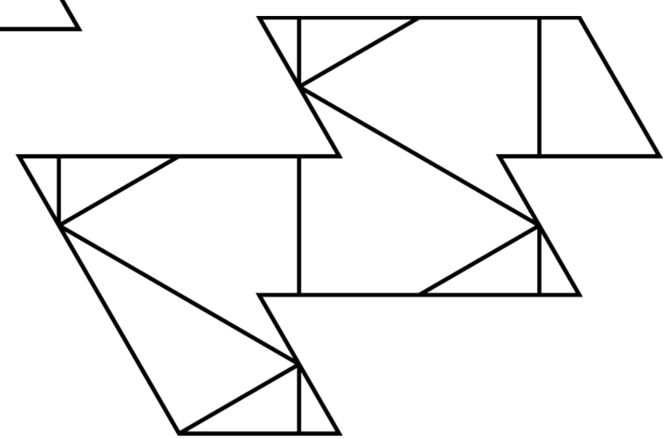
- J84 (Snub disphenoid)











By the way,

- Number of edge unfolding of JZ solids [Horiyama, Shoji 2013]
  - J01 8
  - J02 16
  - J03 308
  - J04 3,030
  - J05 29,767
  - J06 7,825,005
  - …
  - J71 2,079,942,317,394,110,986,896,181,956,672
  - J72 20,668,673,558,050,742,614,946,330,896
  - J73 10,597,511,106,353,370,064,654,696,448
  - …

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We cannot check all edge unfolding of them!!

Outline of "filtering"

Obs: Most JZ solids seem to have no edge unfolding that can fold to a regular tetrahedron

- We negatively check edge unfolding of each convex polyhedra;
  1. If it has no edge unfolding tiling, we cannot fold any tetramonohedron.
     - We found previous results in [Akiyama et al. 2011]
  2. If its area and length have no consistency, we cannot fold to a regular tetrahedron.
  3. If some solids pass these tests, we have to try all possible combinations in any way…

Mainly I show 2 and 3.

Two points

- Point 1 :
  - Why always "(**Regular) tetrahedron** v.s. ..." ?

We can use "tiling" property!

- Point 2 :
  - How to obtain **edge unfolding** ?

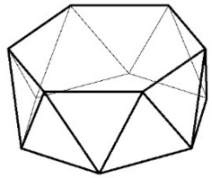
We need some (non-trivial) algorithm…

Edge unfolding which tiles

1. Some edge unfolding can be tiling for [Akiyama et al. 2011]:
   - JZ Solids (18): J1, J8, J10, J12, J13, J14, J15, J16, J17, J49, J50, J51, J84, J86, J87, J88, J89, J90
   - Regular 6 anti-prism

Note 2: Luckily, they consist of only regular triangles and squares.

Note 1: Actually, in [Akiyama et al 2011], they consider “tiling”, but we have to consider p2 tiling.



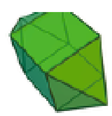
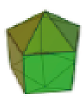
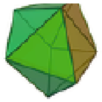
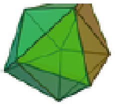
Other solids have no unfolding that can tile the plane

2. Check the consistency of “length” and “area” of the solids and a regular tetrahedron…

Check the consistency of “length” and “area”

- For each Jx that has tiling unfolding,
  1. Find its area Sx
  2. Find the edge length Lx of a regular tetrahedron of area Sx
  3. Consider a partial tiling by △s and □s that form Jx
  4. Lx should be made by two grid points on this partial tiling.

I skip some lemmas…

Name	J15	J16	J17	J49	J50	J51
Image						
# of □s	4	5	0	2	1	0
# of △s	8	10	16	6	10	14
L_{J_i}	$\sqrt{\frac{4\sqrt{3}}{3}+2}$ $= 2.075\cdots$	$\sqrt{\frac{5\sqrt{3}}{3}+\frac{5}{2}}$ $= 2.320\cdots$	2	$\sqrt{\frac{2\sqrt{3}}{3}+\frac{3}{2}}$ $= 1.629\cdots$	$\sqrt{\frac{\sqrt{3}}{3}+\frac{5}{2}}$ $= 1.754\cdots$	$\sqrt{3.5}$ $= 1.870\cdots$

Except J17, J84…

- For each Jx that has tiling unfolding,
  3. Consider a way by filling △s & □s that may fold to Jx, which we called "partial development"
  4. On this partial development, we check if there are a pair of grid points that makes Lx
  5. They make a simple linear summation of $\sqrt{2}$ and $\sqrt{3}$. Thus we cannot make, e.g., $\sqrt{\frac{4\sqrt{3}}{3}+5}$. Thus it cannot fold to that Jx.

Last survivors: J17, J84, with J12, J13, J14, J51, J89

Except J17, J84…

- For each Jx in J12, J13, J14, J51, J89,
  3. Enumerate a way by filling △s & □s that may fold to Jx.
  4. On this partial development, we check if there are a pair of grid points that makes Lx

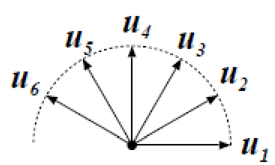
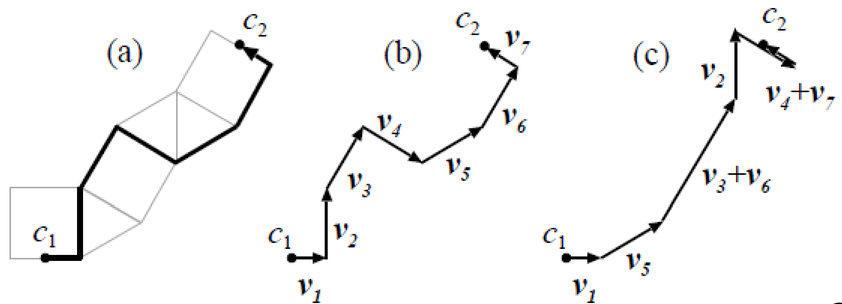


Figure 10: Six unit vectors.



Conference version

For J12, J13, J14, J51, J89, this check took 10 hours…

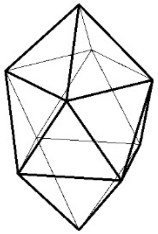
After 1 year, I found that when I wrote a journal paper

Using the vectors, since they can be “sorted”, the computation time become less than 1 second!!

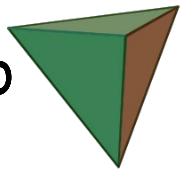
Journal version

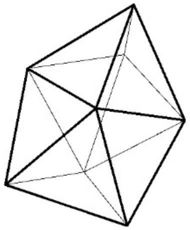
Our Results

- Johnson-Zalgaller solids : 92 solids

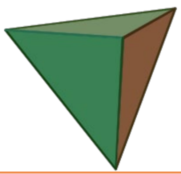


- **J17** (Gyroelongated square dipyramid) 13,014 edge unfolding
  - 78 unf. : 1 way
  - 8 unf. : 2 ways
  - 1 unf. : **3 ways**

  } of folding into 



- **J84** (Snub disphenoid) 1,109 edge unfolding
  - 32 unf. : 1 way
  - 5 unf. : **2 ways**

  } of folding into 

For J17 and J84, we

1. First enumerate all ways of edge unfolding (by Horiyama-san who is an expert for using a data structure BDD).
2. Since they are not so HUGE, we check each edge unfolding if it is a p2 tiling.

… <u>How can we check it?</u>

Folding problem

- In general, the following folding problem is not easy:

Input: Simple polygon P and a convex polyhedron Q

Output: Determine if P can fold to Q

(and we like to have crease lines if P can fold to Q).

We have no general algorithm so far…

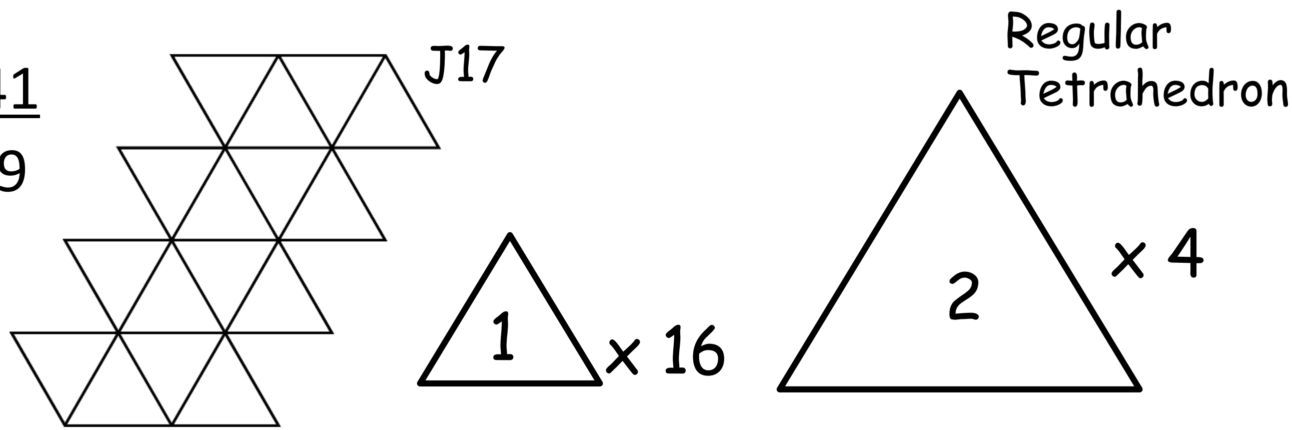
The our problem has nice properties;

- P consists of regular triangles
- Q is a regular tetrahedron

Hence our algorithm checks if P is a p2 tiling.

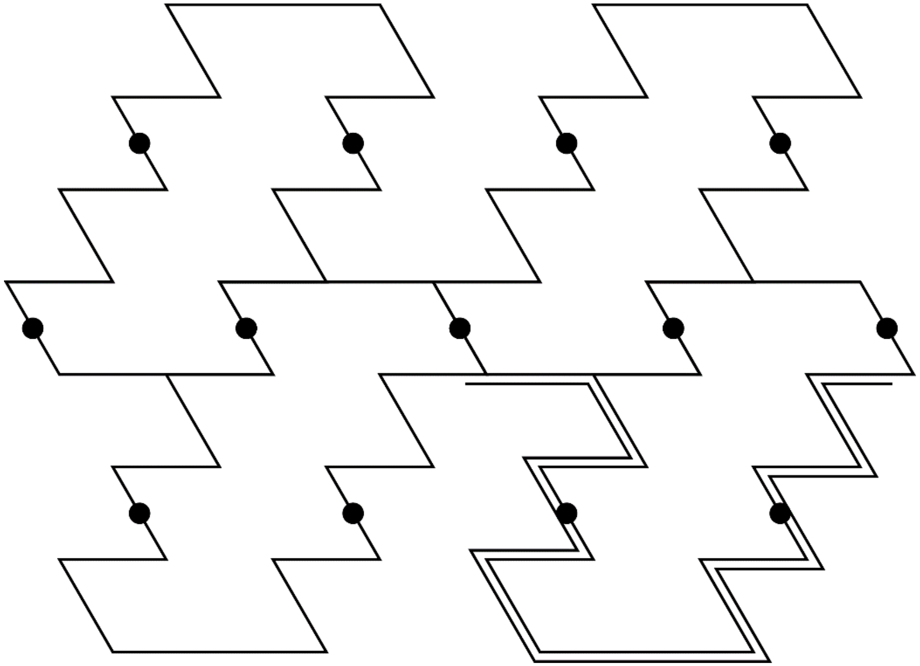
Algorithm for J17, J84

- Enumeration of edge unfolding is done [Horiyama, Shoji 2011, 2013]
- We have
  - J17 13,041
  - J84 1,109



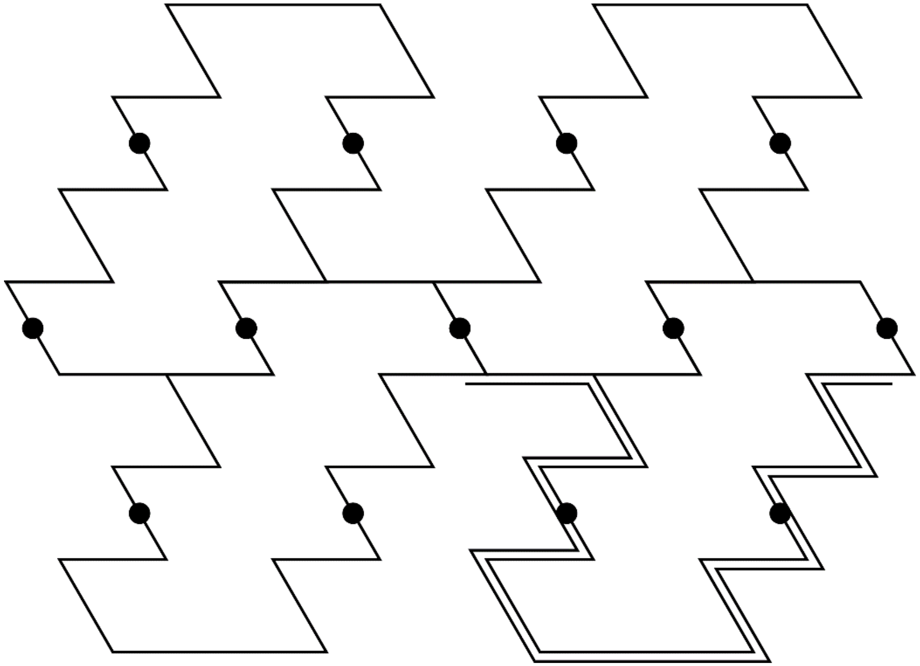
- Cyclic list keeps inner angle at each vertex
  - (and central point of length ½ for gluing) — I omit in this class
  - 60, 180, 120, 180, 180, 180, 60, 180, 300, 180, 60, 180, 300, 180, 60, 180, 300, 180, 60, 180, 120, 180, 180, 180, 60, 180, 300, 180, 60, 180, 300, 180, 60, 180, 300, 180
- Guess two rotational centers r_1, r_2, and check $r_1 r_2 = 2$

Algorithm for J17, J84



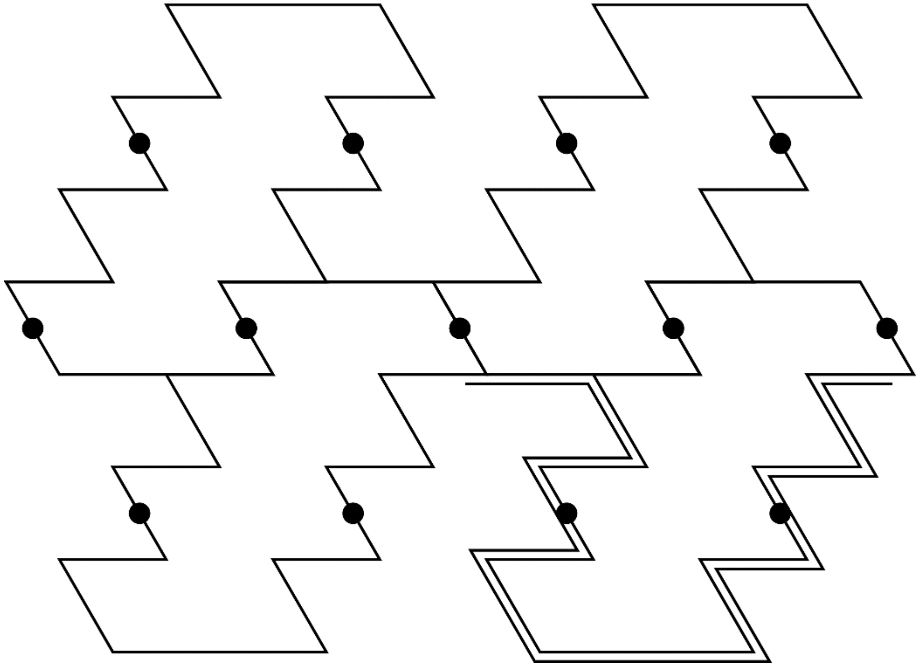
- 60, 180, 120, 180, 180, 180, 60, 180, 300, 180, 60, 180, 300, 180, 60, 180, 300, 180, 60, 180, 120, 180, 180, 180, 60, 180, 300, 180, 60, 180, 300, 180, 60, 180, 300, 180
- For the two rotational centers r_1, r_2, rotate 180° at r_1, r_2

Algorithm for J17, J84



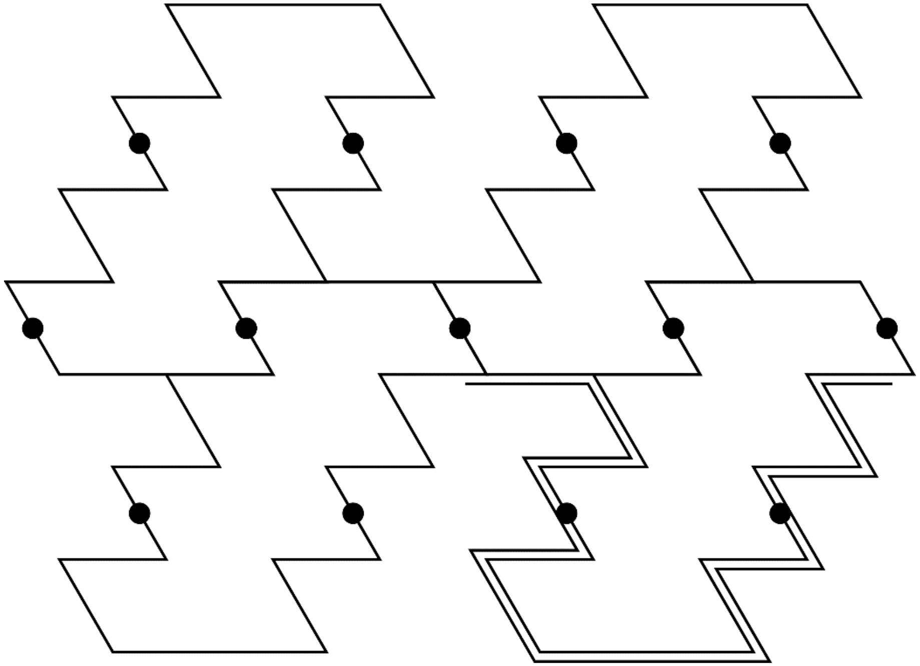
- 180, 180, 240, 180, 300,
  180, 60, 180, 300, 180, 60, 180, 120, 180, 180, 180, 60, 180,
  300, 180, 60, 180, 300, 180, 60, 180, 300, 180
- For the two rotational centers r_1, r_2, rotate 180° at r_1, r_2

Algorithm for J17, J84



- 180, 180, 240, 180, 300,
  180, 60, 180, 300, 180, 60, 180, 120, 180, 180, 180, 60, 180,
  300, 180, 60, 180, 300, 180, 60, 180, 300, 180
- Guess the third rotational center r_3, rotate 180° at r_3, and find r_4

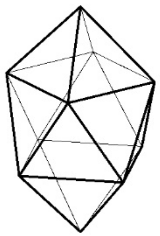
Algorithm for J17, J84



- Guess the third rotational center r_3, rotate 180° at r_3, and find r_4
- Determine if r_1, r_2, r_3, r_4 form a regular triangular lattice or not.

Summary

- Johnson-Zalgaller solids : 92 solids



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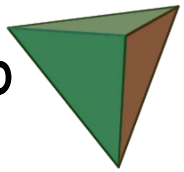
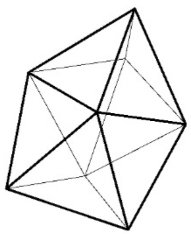
    - 1 unf. : **3 ways**

    of folding into



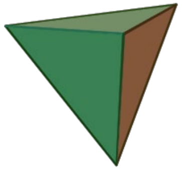


  - **J84** (Snub disphenoid) 1,109 edge unfolding
    - 32 unf. : 1 way
    - 5 unf. : **2 ways**

    of folding into

- Archimedean solids
- *n*-gonal prism
- *n*-gonal antiprism
- other 90 Johnson-Zalgaller solids

No common unfolding with



Open problems (or tough exercises)

- How about other solids except a regular tetrahedron?
- Existence/non-existence of common unfolding
- Edge unfolding/general unfolding
- Few days ago, Kamata-kun and Uehara found that some pairs of solids can be transformed with each other by “twisting”. Can they have common unfolding?