Computational Origami

Ryuhei Uehara
Japan Advanced Institute of Science and Technology (JAIST)
School of Information Science
uehara@jaist.ac.jp
http://www.jaist.ac.jp/~uehara

Today's Topic

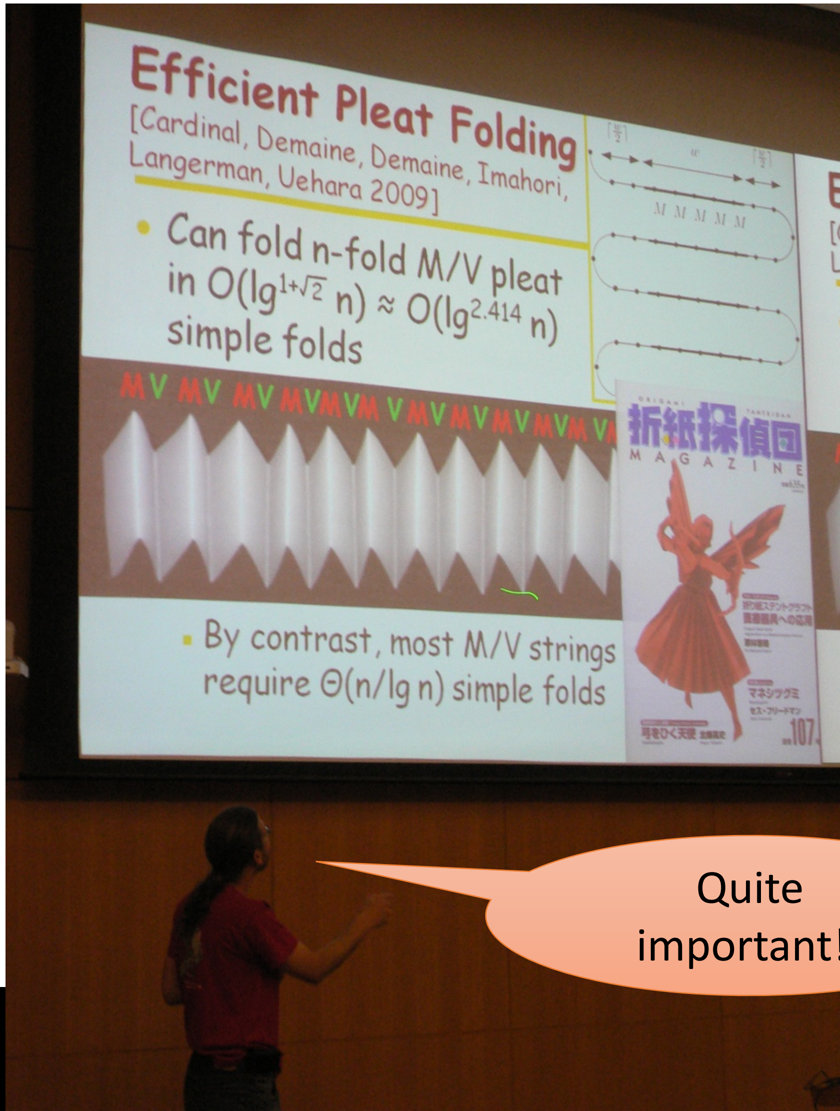
5. Time Complexity
    - "Folding complexity"
        - Theoretically, the world fastest algorithm for pleat folding
    - We can use some techniques in TCS.
        - Recursive analysis and Fibonacci sequence
        - Lower bound by counting method
6. Space Complexity (?)
    - Stamp Folding Problem
    - Minimization of Crease width
        - NP-complete problem, FPT algorithm
7. Undecidable Origami Problem
    - Diagonalization and undecidability

Stamp Folding Problem

Reference:

Ryuhei Uehara, Stamp foldings with a given mountain-valley assignment in *ORIGAMI*[5], pp. 585-597, CRC Press, 2011.

• Pleat folding is…



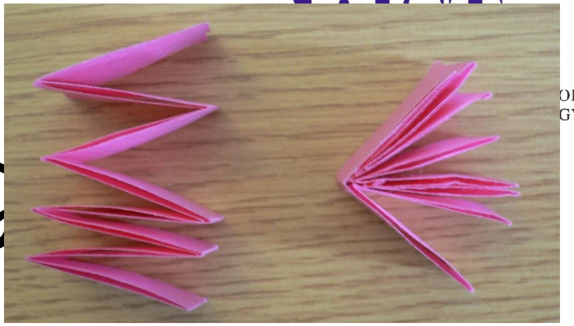
Quite important!!

- Alternating foldings of *Mountain* and *Valley*
- Basic tool of Origami
- Many applications
- ➢ Extension to General Patterns and consider its complexity…

Tokyo Monorail

JAIST Bus

• Complexity of folding



- From the viewpoint of Computer Science
- Two resources of a computation model;
  1. time
  2. space

Crease width of a paper＝
the number of papers between two hinged papers

- Two *resources* of Origami model?
  1. time…the number of foldings (operations)
     - J. Cardinal, E. D. Demaine, M. L. Demaine, S. Imahori, T. Ito, M. Kiyomi, S. Langerman, R. Uehara, and T. Uno: Algorithmic Folding Complexity, *Graphs and Combinatorics*, accepted, 2010.
  2. space…???

It is better to have *less stretch*
at each crease point

New problem

Crease width minimization problem

Input: Paper of length $n+1$ and $s \in \{M, V\}^n$

Output: folded paper according to s

Goal: Find a *good* folded state with small *crease width*

- At each crease, the number of papers between the papers hinged at the crease is *crease width.*
- Two minimization problems;
  - minimize maximum
  - minimize total (=average)

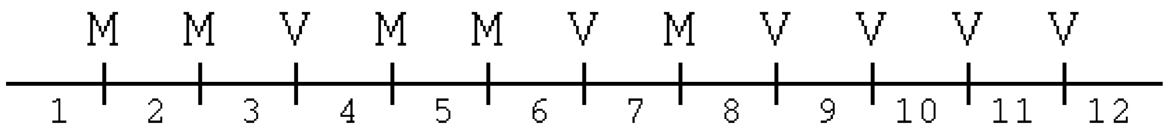
No!!

It seems simple,
… so easy??

New problem

- Simple non-trivial example

Input: MMVMMVMVVVV

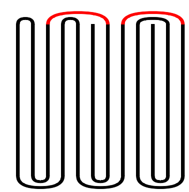


The number of feasible folded states : **100**

Goal: Find a *good* folded state with small *crease width*

- The **unique** solution having min. max. value 3

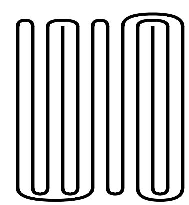
[5|4|3|6|7|1|2|8|10|12|11|9]



total=13

- The **unique** solution having min. total value 11

[5|4|3|1|2|6|7|8|10|12|11|9]



Max=4

New problem

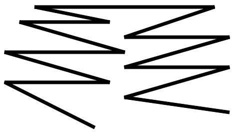
- Minimization problem for **crease width**

Input: Paper of length $n+1$ and $s \in \{\mathrm{M}, \mathrm{V}\}^n$

Output: folded paper according to s

Goal: Find a *good* folded state with small *crease width*

- Two minimization problems; max/total(average)
- A few facts;
  - a pattern has a unique folded state iff it is pleats
  - solutions of {min max} and {min total} are different depending on a crease pattern.
  - there is a pattern having exponential combinations



Minimization of crease width

- Open problem : (Solved later…)
  - Tractable/Intractable?
    - NP-hard?
    - Poly-time solvable by Dynamic Programming?

    It is NP-hard in general for min-max, and still open for total.

    If total is k, we find a fixed parameter tractable algorithm for k.
    - Then, exhaust search technique, that produces ***all*** possible folding ways, works?

average?

exponential pattern ⟷ pleats

No!!

Average case…

The number of folding ways for a *random* pattern

- $\Theta(1.65^n)$ by experiments
- $\Omega(1.53^n)$ and $O(2^n)$ by theoretical lower/upper bounds

… so a naïve program runs *veeeerrrrry* slow.

This problem is known as a stamp folding problem, which is a classic open problem, and I accidentally improved the lower bound ☺

Stamp folding problem

Average number of folding ways for a random pattern = $f(n)/2^n$, where

$f(n) =$ # of folding ways (or folded states) of a paper of length $n+1$

(summation for all possible patterns)

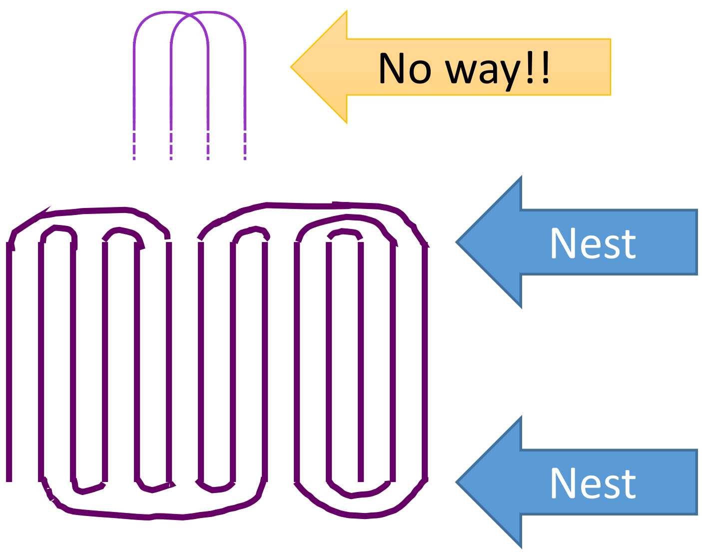
- I give some bounds of $f(n)$:
  - "The On-Line Encyclopedia of Integer Sequences" tells us up to n=28 (by *enumeration*); $f(n) \sim \Theta(3.3^n)$.
  - I have upper/lower bounds;
    - upper bound: $f(n)=O(4^n)$
    - lower bound: $f(n)=\Omega(3.07^n)$

You should know OEIS

Stamp folding problem

The upper bound $O(4^n)$ comes from the Catalan number.

[Proof] If the paper of length $n+1$ is folded,
the crease points should be nested.



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Combination of $n/2$ pairs of () = Catalan Number $C_{n/2}$

×

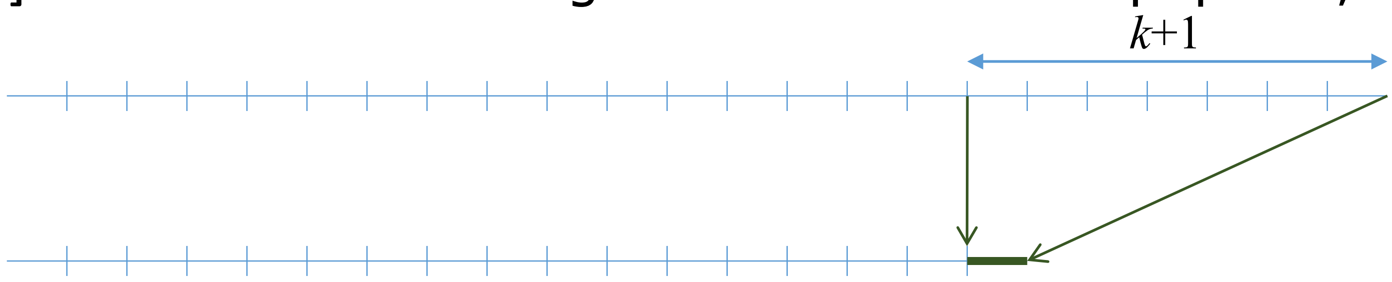
Combination of $n/2$ pairs of () = Catalan Number $C_{n/2}$

Stamp Folding Problem

Lower bound

[Theorem] Its lower bound is $\Omega(3.07^n)$.

[Proof] We consider "folding of the last $k+1$ unit papers";

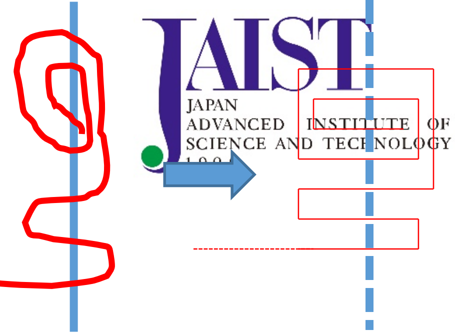


We let

- $f(n)$: the number of folding ways of length $n+1$
- $g(k)$: the number of folding ways of length $k+1$ s.t.
  the leftmost endpoint is not covered

Then, we have $$f(n) \geq (g(k))^{\frac{n}{k-1}} = \left(g(k)^{1/(k-1)}\right)^n$$

Stamp Folding Problem



- Lower bound

[Theorem] Its lower bound is $\Omega(3.07^n)$.

[Proof] We consider "folding of the last $k+1$ unit papers";

$g(k)$: the number of folding ways of length $k+1$ s.t. the leftmost endpoint is not covered

is equal to "the number of ways a semi-infinite directed curve can cross a straight line k times",

A000682 in "The On-Line Encyclopedia of Integer Sequences".

From that site, we have $g(43)=830776205506531894760$.

Thus, by

$$f(n) \geq (g(k))^{\frac{n}{k-1}} = \left(g(k)^{1/(k-1)}\right)^n$$

we have the lower bound.

also obtained by enumeration

Open problem and Future work

Minimization of crease width is basically settled;

- Erik D. Demaine, David Eppstein, Adam Hesterberg, Hiro Ito, Anna Lubiw, Ryuhei Uehara and Yushi Uno: Folding a Paper Strip to Minimize Thickness, *WALCOM 2015*, Lecture Notes in Computer Science Vol. 8973, pp. 113-124, 2015/02/26-2015/02/28, Dhaka, Bangladesh.
- Takuya Umesato, Toshiki Saitoh, Ryuhei Uehara, Hiro Ito, and Yoshio Okamoto: Complexity of the stamp folding problem, Theoretical Computer Science, Vol. 497, pp. 13-19, 2012.

- Some unsettled problems;
  - Minimization of crease width for “total” model

For stamp folding problem;

- What is the most complex pattern of M/V that has the most feasible folded states?

Extension of Stamp Folding

2 dimensional → Map Folding Problem

Input: 2 dimensional (orthogonal) M/V pattern

Output: Does it have folded state of size 1 × 1?

- The complexity of map of size m × n is famous open problem
- For map of size 2 × n, $O(n^9)$ time algorithm was proposed , but…
- Difficult example;