September 19, 1996 Report No. 4

NP-complete problems on a 3-connected cubic planar graph and their applications

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Abstract

It is shown that the vertex cover problem (or the maximum independent set problem) remains NP-complete even for a cubic, planar, and 3-connected graph of girth greater than 3. The result adds the restrictions of the 3-connectedness and the girth greater than 3 to the previously known result. New NP-complete problems for faces on a triangulation of the sphere are produced.

Key words: NP-complete problems, Triangulations of the sphere.

1 Introduction

It is well known that finding a maximum independent set (MIS), or a vertex cover (VC) of a given graph is NP-complete even for a cubic planar graph [2]. We show that the VC problem (or the MIS problem) remains NP-complete even for a cubic, planar, and 3-connected graph of girth greater than 3. We add the 3-connectedness and the restriction on the girth to the previously known result. The 3-connectedness of a planar graph is an important property, since a graph forms the edge structure of a convex polyhedra if and only if it is planar and 3-connected ([3, 1]). Das and Goodrich showed that several well-known computational geometry problems involving 3-dimensional convex polyhedra are NP-complete. To show the completeness, they used two NP-complete problems, that is, the VC problem on a 3-connected planar graph, and the VC problem on a c-stellation of a 3-connected cubic planar graph for any constant $c \geq 4$. Since our problem is contained by two NP-complete problems used by Das and Goodrich above, our result simplifies their proofs.

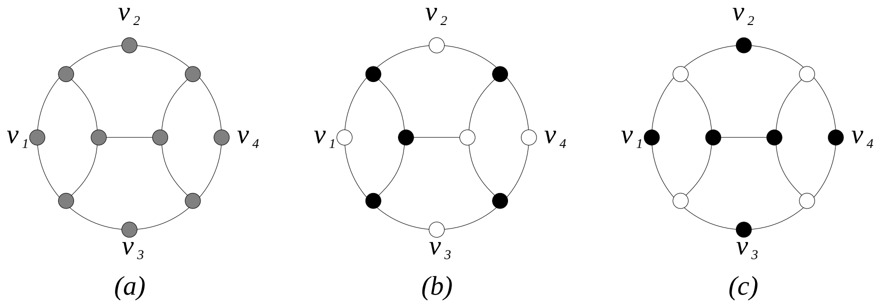

Figure 1: Gadget and its VCs

A triangulation is widely investigated and it has many applications (see [5] for example). The geometric dual graphs of a 3-connected cubic planar graph is a triangulation of the sphere. For a 3-connected cubic planar graph G and its geometric dual graph G^*, a vertex of G corresponds to a face of G^*. Thus the problems for vertices of G correspond to the problems for faces of G^*. Using this, we have NP-complete problems for faces on a triangulation of the sphere.

2 Main results

2.1 NP-completeness of the problems on a restricted graph

First, we show a lemma for the connectedness of a cubic graph.

Lemma 1 *A cubic graph is 3-connected if and only if it is 3-line-connected.*

Proof. By definition, 3-connected graph is 3-line-connected for any graph. Thus we show that 3-line-connected cubic graph is 3-connected. Let $G = (V, E)$ be a 3-line-connected cubic graph. Then there exist three edges e_1, e_2, and e_3 whose removal results in a disconnected graph. It is sufficient to show that each two of e_1, e_2, and e_3 share no common vertex. Assume that e_1 and e_2 have a common vertex v for a contradiction. Since the degree of v equals three, the other edge e_4 incident to v exists. If $e_3 = e_4$, it is easy to see that G is 1-line-connected for some e_i with $1 \leq i \leq 3$. Otherwise, G is 2-line-connected for $\{e_3, e_4\}$ since the degree of v equals three. In each case, it contradicts that G is 3-line-connected. This implies that e_1 and e_2 share no common vertex. By symmetric, e_1, e_2, and e_3 share no common vertex. Let S be a set of one of endpoints of e_i with $1 \leq i \leq 3$. Then clearly, S witnesses that G is 3-connected. ∎

For a "gadget" shown in Figure 1(a), the following lemma holds.

Lemma 2 *Let $U = \{v_1, v_2, v_3, v_4\}$. Then (1) the minimum VC of the gadget has size equal to 5, and it contains no vertex in U; (2) for every VC on the gadget, if it contains at least one vertex in U, it has size greater than 5, and (3) there exists a VC of size equal to 6, which contains every vertex in U.*

Proof. We must select at least four vertices to cover eight edges around the gadget, and at least one of two center vertices to cover center edge. Thus every VC contains at least five vertices. Thus, the VC of size five is only one type shown in black in Figure 2(b), which contains no vertex in U. Now, it is not difficult to see that if a VC contains at least one vertex in U, it has size greater than 5. The VC shown in black in Figure 1(c) completes the lemma. ∎

Reminding that the *girth* of a graph G is the length of a shortest cycle in G, we are ready to show main theorem in this section.

Theorem 3 *The following problem is NP-complete: For a given 3-connected cubic planar graph G of girth greater than 3 and a positive integer k, is there a VC of size k?*

Proof. We provide a polynomial time reduction from the following known NP-complete problem [2]: For a given cubic planar graph $G_0 = (V_0, E_0)$ and positive integer k_0, is there a VC of size k? For a given cubic planar graph G_0, let S_0 be a set of each edge whose removal results in a disconnected graph. For each

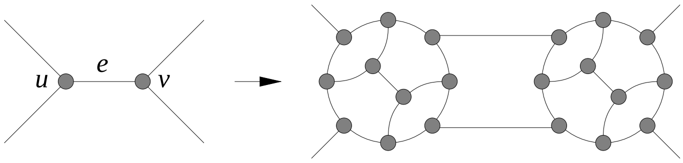

Figure 2: Replacement an edge by two gadgets

$e = \{u, v\} \in S_0$, replace it by two gadgets as shown in Figure 2. Let $G_1 = (V_1, E_1)$ be the resulting graph. Then, clearly, G_1 is a 2-line-connected cubic planar graph. Here we consider the following claim: G_0 has a VC of size k_0 if and only if G_1 has a VC of size $k_0 + 10 \mid S_0 \mid$. To prove the claim, let $e = \{u, v\} \in S_0$, and $C \subset V_0$ be the minimum VC of G_0. Since C is the VC, it contains u or v. By Lemma 2, if C contains both u and v, the minimum VC of G_1 must contain at least 12 vertices to cover two gadgets corresponding to u and v; otherwise, 11 vertices are sufficient to cover two gadgets. This establishes the claim.

Next, for the 2-line-connected cubic planar graph $G_1 = (V_1, E_1)$, let S_1 be a set of each pair of edges of E_1 whose removal results in a disconnected graph. For each pair in S_1, select one of them, and replace it by two gadgets as shown in Figure 2. Let $G_2 = (V_2, E_2)$ be the resulting graph. Then, clearly, G_2 is a 3-line-connected cubic planar graph. By Lemma 1, G_2 is a 3-connected cubic planar graph. We can show the following claim by the same way as the first claim: G_1 has a VC of size k_1 if and only if G_2 has a VC of size $k_1 + 10 \mid S_1 \mid$.

For each cycle of length 3, selecting an arbitrary edge of the cycle, and replacing the edge by two gadgets in the same way, we have the graph G_3 of girth greater than 3. The same arguments as the claims above completes the proof. ∎

For any graph $G = (V, E)$, $V' \subseteq V$ is an MIS of G if and only if $V - V'$ is a VC of G [2]. This implies the following corollary.

Corollary 4 *The following problem is* NP*-complete: For a given 3-connected cubic planar graph G of girth greater than 3 and a positive integer k, is there a MIS of size k?*

Das and Goodrich proved NP-completeness of the following two problems; the VC problem on a 3-connected planar graphs, and the VC problem on a c-stellations of a 3-connected cubic planar graph for any constant $c \geq 4$. Here, a *stellation* of a face f in G as the insertion of a vertex in the interior of f that we then make adjacently to each vertex of f. A c-stellation of a graph is the result of performing c consecutive stellations on the graph (see [1], for detail). These VC problems contain the VC problem in Theorem 3. Thus, using the theorem, we can simplify the proofs in [1].

2.2 NP-complete problems on a triangulation of the sphere

Whitney expressed planarity with the existence of dual graphs. Given a plane graph G, its *geometric dual graph* G^* is constructed as follows; place a vertex in each face of G (including the exterior region) and, if two regions have an edge e in common, join the corresponding vertices by an edge e^* crossing only e. The geometric dual graph of a 3-connected graph is always a simple graph (see [3] for detail). A *triangulation* of the sphere is a planar graph such that every face of the graph, including the exterior region, is a triangle. We first show the following lemma.

Lemma 5 *Geometric dual graph of a 3-connected cubic planar graph is a triangulation of the sphere.*

Proof. Let $G = (V, E)$ be a 3-connected cubic planar graph and $G^* = (V^*, E^*)$ be its geometric dual graph. Since G is cubic, $3 \mid V \mid= 2 \mid E \mid$ holds. By definition of the duality, $\mid E \mid=\mid E^* \mid$ holds. Since G is connected, Eular's Theorem states that $2+ \mid E \mid - \mid V \mid - \mid V^* \mid= 0$ (see [4, Lecture 14] for example). Combining the equations, we get that $\mid E^* \mid= 3 \mid V^* \mid -6$. Here, G^* is a triangulation of the sphere if and only if $\mid E^* \mid= 3 \mid V^* \mid -6$ (see [4, Theorem 14.10] for example). This completes the proof. ∎

For a given triangulation of the sphere and a positive integer k, we define two problems on them as follows.

Independent triangle set

Question: Are there a triangle set of size k or more such that no two triangles in the set have an edge in common?

Shaping triangulation

Question: Are there a triangle set of size k or less such that every edge of the triangulation is had at least one triangle in the set?

We remark that the shaping triangulation problem is the problem to minimize the number of triangles to shape the triangulation of the sphere. We show the main theorem in this section.

Theorem 6 *Both the independent triangle set problem and the shaping triangulation problem are* NP-*complete.*

Proof. By Lemma 5, the MIS problem and the VC problem on a 3-connected cubic planar graph can be reduced to the independent triangle set problem, and the shaping triangulation problem, respectively. ∎

It is easy to see that the girth of a graph corresponds to the minimum degree of its geometric dual graph. Thus, the independent triangle set problem and the shaping triangulation problem remain NP-complete even if we add the restriction that the minimum degree is greater than 3 to a given triangulation.

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