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HILBERT'S FINITISM, THE CONCEPT OF INFINITY, AND THE DECISION PROBLEM

Abstract: The main focus of my talk is on a critical analysis of some aspects of Hilbert's proof-theoretic programme in the 1920s. During this period, Hilbert developed his metamathematics or proof theory to defend classical mathematics by carrying out, in a purely finitist fashion, consistency proofs for formalized mathematical theories *T*. The key idea underlying metamathematical proofs was to establish the consistency of *T* by means of weaker, but at the same time more reliable methods than those that could be formalized in *T*. It was in the light of Gödel's incompleteness theorems that finitist metamathematics as designed by Hilbert and his collaborators turned out to be too weak to lay the logical foundations for a significant part of classical mathematics. Hilbert responded to Gödel's challenge by extending his original finitist point of view. The extension was guided by two central, though possibly conflicting ideas: firstly, to make sure that it preserved the quintessence of finitist metamathematics; secondly, to carry out, within the extended proof-theoretic bounds, a finitist consistency proof for a large part of mathematics, in particular for second-order arithmetic.

I begin by describing succinctly the main features of Hilbert's early conception of informal, non-axiomatic metamathematics by contrasting it with modern formalized and axiomatized metamathematics, including his crucial distinction between real and ideal statements. The second section forms the core of my talk. Here I discuss what I call "assumptions of infinity". Let *U* be a proposition (or a set of propositions). When we prove *U* or acknowledge its truth, we make an asumption of infinity if and only if *U* says on a contentual interpretation (a) that there is an infinite object or (b) that *U* is satisfiable only in an infinite domain. To make both (a) and (b) more precise, the expression "infinite" calls for explication. However, as far as Hilbert's texts on proof theory are concerned, he does not offer any explicit explication of the term "infinite". This applies especially to his distinguished transfinite axiom "If a predicate A applies to an object τA, then it applies to all objects *a*' which is said to be the primary source of all transfinite concepts, principles, and axioms. I argue that assumptions of infinity do not occur in Hilbert's formalized mathematics, but, rather unexpectedly, that they do underlie his informal metamathematics. I consider three examples: quantification, formalized proofs, and what, following his jargon, we may call "capability of being negated". The fact that in advocating finitist metamathematics in the 1920s Hilbert did not dispense with assumptions of infinity is diagnosed as the Achilles' heel of his appproach. In the third section, I make a number of critical remarks on the extensions of the finitist point of view suggested by Hilbert and Bernays in *Foundations of Mathematics* (vol. I, 1934, vol. II, 1939). I argue, in particular, that in the second volume of this work the old methodological principles of finitism — representability in perceptual intuition and intuitive evidence, surveyability and unassailable soundness — have by and large disappeared, due to pure proof-theoretic pragmatism. Formalized metamathematics embraces here at least Peano Artithmetic (PA), but it could be even as wide as PA plus transfinite induction up to ε_0 (the smallest ε-number of Cantor's second number class $Z(\aleph_o)$, which is the set $\{\alpha\}$ of all order types α of well-ordered sets of cardinality $\aleph_o$).

The concluding section is devoted to remarks on the so called decision problem and its relation to Hilbert's finitist approach. Quite generally speaking, the problem of finding a decision procedure or an algorithm for deciding whether an arbitrary well-formed formula of a logical theory *T* is a theorem of *T* is called the decision problem. A positive solution is a proof that such a decision procedure exists (such is the case for the propositional calculus). A negative solution is a proof that there can no such procedure (as in the case of the first-order predicate calculus with identity). In my account, I rely on original texts written by Hilbert and Ackermann (1928), Church (1936) and Turing (1936). The situation around 1935 was that an exactly defined class *C* of computable number-theoretic functions considered by Church and Kleene, called the "λ-definable functions", had been found to possess properties suggesting that *C* might comprise all functions that can be regarded as computable from the point of view of our intuitive notion of computation. Another class of computable functions, the "general recursive functions", first defined by Gödel, had similar properties. In 1936, Church proved that the two classes coincide and advanced the thesis (Church's thesis) that all effectively calculable (computable) functions are λ-definable or general recursive. A little later but independently, Turing introduced another precisely defined class of intuitively computable functions (which may be called "Turing computable functions") and made the same claim for this class: Turing's thesis. Turing showed that his computable functions are the same as the λ-definable functions, and hence the same as the general recursive functions, from which it follows that Turing's and Church's theses are equivalent.