Some/Any is Easy, Effective, and Necessary (A Brief Overview)

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Abstract

We address a Some/Any property that plays a crucial role in proving formally the decidability of α-equivalence in the λ-calculus. The property enables extensionality, transitivity, and computability and is widely relevant, not least because of its *zero-knowledge* proof. See [1] for details.

Let $\overset{\overrightarrow{x_i}}{\dashrightarrow}_{i\alpha_0^{\mathrm{io}}}$ be the function that renames every abstracted variable in a λ-term, Λ^{var}, using the names in the vector $\overrightarrow{x_i}$ in a left-to-right order.

Definition 1 $\Lambda^{\mathrm{var}} ::= \mathcal{VN} \mid \Lambda^{\mathrm{var}}\Lambda^{\mathrm{var}} \mid \lambda\mathcal{VN}.\Lambda^{\mathrm{var}}$

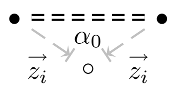
The function can be defined such that it is guaranteed-by-construction to be computable whenever the $\overrightarrow{x_i}$-vector consists of the right amount of fresh variable names, i.e., variable names that do not occur in the term being reduced.

Abstractly speaking, the set $\mathcal{VN}$ that is used to define Λ^{var} is a one-sorted set of variable names such that the following holds.

Assertion 2 *$\mathcal{VN}$-equality, $=_{\mathcal{VN}}$, is decidable.*

Assertion 3 *There exists a total, computable function on finite subsets of $\mathcal{VN}$,* $\mathrm{Fresh}(-) : \mathcal{P}_{\mathrm{fin}}(\mathcal{VN}) \longrightarrow \mathcal{VN}$*, such that:* $\mathrm{Fresh}(\mathrm{VN}) \notin \mathrm{VN}$.

When trying to prove the decidability of α-equivalence, we see that we can use $\overset{\overrightarrow{z_i}}{\dashrightarrow}_{i\alpha_0^{\mathrm{io}}}$, with fresh $\overrightarrow{z_i}$ to reduce α-equivalence to syntactic equality (which is decidable further to Assertion 2) in the following manner.



Unfortunately, any initial attempt to establish the diagram fail.

Failing extensionality: Consider $\lambda x.e ==_\alpha \lambda x.e'$. By I.H. the diagram exists for e and e' using, say, $\overrightarrow{z_i}$. When addressing the abstractions using, e.g., $z\,\overrightarrow{z_i}$, we get stuck as $\overrightarrow{z_i}$ can conflict with x (and this *is* a real problem).

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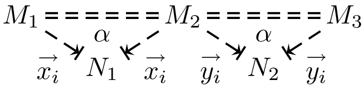
Failing transitivity: By I.H., we have the following in the transitive case.

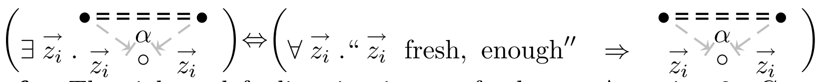


If $\overrightarrow{z_i}$ in the considered diagram is existentially quantified, $\overrightarrow{x_i}$ and $\overrightarrow{y_i}$ above need not be identical. If $\overrightarrow{z_i}$ is universally quantified over fresh names, $\overrightarrow{x_i}$ and $\overrightarrow{y_i}$ are also fresh with respect to M_2 (not just M_1 and M_3).

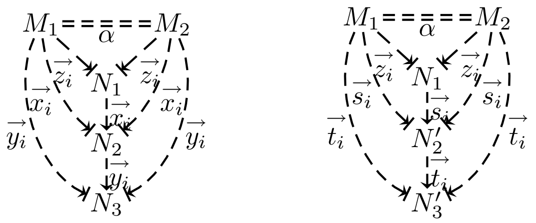
Failing computability: Unless we can establish the diagram with universal quantification (which seemingly is harder than existential quantification), we cannot guarantee that there is a computable choice of $\overrightarrow{z_i}$ that will work.

Surprisingly, existentially and universally quantifying $\overrightarrow{z_i}$ are equivalent.

Lemma 4



Proof The right-to-left direction is easy further to Assertion 3. Consider, therefore, the other direction and grant us that $\overset{\overrightarrow{x_i}}{\dashrightarrow}_{i\alpha_0^{\mathrm{io}}}$ is transitive.



N_1 and $\overrightarrow{z_i}$ exist by assumption. Introducing the N_2 (on the left) strengthens the "for some" $\overrightarrow{z_i}$ to "for any" $\overrightarrow{x_i}$ that are fresh with respect to N_1, M_1, and M_2. Unfortunately, the variables in N_1 are still excluded from consideration. Constructing the N_2' on the right allows us to use variables, $\overrightarrow{s_i}$, that are fresh with respect to any specific $\overrightarrow{x_i}$ as well as N_1, M_1, and M_2. By subsequently adding the layers of N_3 and N_3', we can on the one hand use any $\overrightarrow{y_i}$ that are fresh with respect to M_1, M_2, and any specific $\overrightarrow{x_i}$. On the other hand, we can also use any $\overrightarrow{t_i}$ that are fresh with respect to M_1, M_2, and any specific $\overrightarrow{s_i}$. As $\overrightarrow{s_i}$ are fresh with respect to any specific $\overrightarrow{x_i}$ by construction, we are thus able to use any variable names, $\overrightarrow{y_i}$ or $\overrightarrow{t_i}$, that are fresh with respect to just M_1 and M_2. □

Theorem 5 *α-equivalence is decidable by means of $\dashrightarrow_{\alpha_0^{\mathrm{io}}}$.*
Proof By using Lemma 4 all problems go away. □

[1] René Vestergaard. *Some/Any is Easy, Effective, and Necessary (A Formal Proof of the Decidability of α-Equivalence).* Manuscript. Avaiable at http://www.jaist.ac.jp/˜vester.