Stochastic Modeling of Immersed Rigid-body Dynamics

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Abstract

The simulation of immersed rigid-body dynamics involves the coupling between objects and turbulent flows, which is a complicated task in computer animation. In this paper, we propose a stochastic model of the dynamics of rigid bodies immersed in viscous flows to solve this problem. We first modulate the dynamic equations of rigid bodies using generalized Kirchhoff equations (GKE). Then, a stochastic differential equation called the Langevin equation is proposed to represent the velocity increments due to the turbulences. After the precomputation of the Kirchhoff tensor and the kinetic energy of a synthetic turbulence induced by the object moving, we utilize a fractional-step method to solve the GKE with vortical loads of drag and lift dynamics in runtime. The resulting animations include both inertial and viscous effects from the surrounding flows for arbitrary geometric objects. Our model is coherent and effective to simulate immersed rigid-body dynamics in real-time.

CR Categories: I.3.7 [Computer Graphics]: Three-Dimensional Graphics and Realism—Animation;

Keywords: rigid body, turbulence model, generalized Kirchhoff equations, stochastic differential equation, real-time

Figure 1: *A gliding paper airplane simulated by our stochastic modeling approach. The left-bottom subfigure is a contrastive trajectory captured from a video.*

1 Introduction

Immersed rigid-body dynamics considers the motions of rigid bodies which are fully immersed in air or submerged under water. We cannot simulate the dynamics by a conventional rigid-body simulations because they haven't considered the influences from the surrounding turbulent flows. This issue is also distinguishing from two-way coupling animations, which have to apply high computation cost for fluid simulations and seldom consider the coupling with turbulent flows. This research is significant to enhance the simulations of daily-life natural phenomena in various applications, because we are easily conscious of the simulations artefacts. For example, a paper airplane is not always gliding straightly depending on people's experiences as shown in Figure 1.

Two-way coupling is related to our work and has been studied extensively in computer animation. Mainly there are two types of schemes on the coupling animations. The first scheme handles fluid in Euler formulation and rigid bodies in Lagrangian formation. Carlson et al. [Carlson et al. 2004] treated the rigid body as fluid grid by using a distributed Langrange multiplier. Guendelman et al. [Guendelman et al. 2005] proposed a robust ray casting algorithm for the coupling between fluid and cloths to avoid fluid leaking. The second scheme is a fully Langrangian meshless method. Solenthaler et al. [Solenthaler et al. 2007] utilized a penalty method to analyze the forces on the immersed boundary. Becker et al. [Becker et al. 2009] proposed a direct forcing method in a predictor-corrector scheme with SPH particles. All these schemes focus on the motion of the surrounding fluids and requires high computation cost.

For a turbulent flow, the vortices are generated around the body and then detached from the body surface as vortex shedding. The dynamics of rigid bodies becomes unsteady into chaotic motions. There are two types of effects induced by the surrounding fluid: inertial and viscous effects. Regarding the inertial effect, Weissmann et al. [Weissmann and Pinkall 2012] introduced a Kirchhoff tensor into computer graphics for represent added-mass tensors due to the displacement of fluid. This approach is only suitable for the inviscid and irrotational flow with a low Reynolds number, and chose the drag calculation to be linearly related to the body velocity. In this paper, we propose a stochastic model related to the turbulent flow for solving the vortical loads of both drag and lift forces. To interpret the viscous effect, we need to consider the turbulent flow around rigid bodies. In CG literature, synthetic turbulence is a successful way to simulate turbulences. Pfaff et al. [Pfaff et al. 2009; Pfaff et al. 2010] utilized turbulent energy model to give fine details of turbulence simulation, and Chen et al. [Chen et al. 2011] further enhanced the simulation details by stochastic particles. These methods focus on the turbulent performances rather than coupling motion which is significant in this paper.

Recent work [Yuan et al. 2011; Selino and Jones 2013] adopted synthetic turbulence approach for wind effects to simulate the swaying trees and floating leaves. Similar to our work, they computed the kinetic energy of turbulence in an offline step. However, both of these methods simulated bodies as particles where the inertial effect and the coupling between translational and

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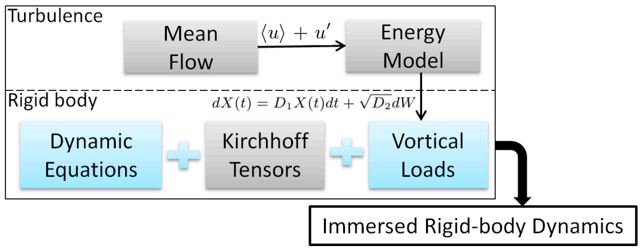


Figure 2: *Overview of our stochastic model. The steps in grey color run in precomputation steps. The runtime includes only two steps in blue color so that the computation cost is significantly reduced.*

rotational velocities haven't been taken into account for immersed rigid-body dynamics.

In this paper, our stochastic model of immersed rigid-body is composed of two components: rigid body dynamics and turbulent energy model as shown in Figure 2. The rigid-body dynamics is defined as the ordinary differential equations (ODEs) based on generalized Kirchhoff equations (GKE), which include the computations of Kirchhoff tensor and vortical loads. After obtaining the kinetic energy and its dissipation rate of turbulent flow, the stochastic differential equations of both translational and rotational velocities are utilized to evaluate the velocity increments. Since the computations of both Kirchhoff tensor and the energy model are in precomputation steps and our model only runs a fractional-step algorithm in runtime to solve the ODEs, the model makes the simulations of immersed rigid-body dynamics feasible and efficient with a low computation cost. The major contributions of this paper are as follows:

- A stochastic model based on the stochastic differential equations of both translational and rotational velocities to represent the velocity increments due to turbulence in flow.
- A fractional-step method to solve GKE with calculated vortical loads due to the viscous effect of the surrounding flow.
- An efficient approach using two precompuationa steps to compute Kirchhoff tensor and turbulent energy model, which usually require a high computation cost.

2 Rigid-body Dynamics

For a rigid body having mass m, center of mass O and uniform density ρ_b moving through a viscous fluid with fluid density ρ_f, the state of the rigid body is described by $(R(t), x(t), v(t), \omega(t))$. $R(t)$ describes the orientation of the body as a 3×3 orthogonal matrix, and $x(t)$ is the position of O in the inertial reference frame at time t. $(v(t), \omega(t)) \in R^6$ are the translational and rotational velocities in body-fixed frame. The kinematic equations of the rigid body are defined in body-fixed frame as follows:

$$\dot{R} = R\hat{\omega}, \qquad \dot{x} = Rv \tag{1}$$

where $R\hat{\omega} = R \times \omega$, and $\hat{\omega}$ is the skew matrix of the rotational velocity ω.

2.1 Generalized Kirchhoff Equations

The dynamics of a rigid body immersed in a viscous fluid results from the coupling between the body and the surrounding fluid. The inertial effect from the interaction of the fluid to a body displacement, including both translational and rotational transformations, is described by added-mass tensors which are called Kirchhoff tensor K in [Weissmann and Pinkall 2012]. Conventionally, the dynamics of immersed rigid body is governed by the generalized Kirchhoff equations [Fernandes et al. 2008]. The GKE has the following ODEs form in the body-fixed frame.

$$\begin{pmatrix} \dot{l} \\ \dot{p} \end{pmatrix} = \begin{pmatrix} l \times \omega + p \times v \\ p \times \omega \end{pmatrix} + \begin{pmatrix} \tau_g \\ f_g \end{pmatrix} + \begin{pmatrix} \tau_t \\ f_t \end{pmatrix} \tag{2}$$

where (l, p) are the translational and rotational momenta, $(l, p)^T = K \cdot (\omega, v)^T$. Because Kirchhoff tensor K is independent of the body's state, K can be approximated by the Neumann boundary conditions in potential flow in a precomputation step [Weissmann and Pinkall 2012]. f_t and τ_t are the resulted force and torque due to the turbulence generated at the body surface while the body moves in a viscous flow; f_g and τ_g result from the buoyancy-corrected gravity. Both the gravity and buoyancy force are applied to the immersed rigid body with inverse directions. We describe them in body-fixed frame as follows:

$$f_g = R^T(m - \rho_f V)g \tag{3}$$
$$\tau_g = \rho_f V\vec{r} \times R^T g \tag{4}$$

where V is the volume of the body and $\vec{r}$ is the vector from the center of mass to the center of buoyancy in body-fixed frame.

The challenge of solving Equation 2 is how to determine the force f_t and torque τ_t due to viscous effect of surrounding turbulent flow, which includes the drag and lift dynamics. Therefore, f_t has the following formation depending on aerodynamics.

$$f_t = f_{drag} + f_{lift} = -\frac{1}{2}\rho_f A|U|(C_d U + C_l U \times \frac{N \times U}{|N \times U|}) \tag{5}$$

where A is the surface area of the body, and U is an intermediate translational velocity which we will explain later. C_d and C_l are the drag and lift coefficients, which can be given from experiments or advanced analysis. Here we let them as user control parameters so that different dynamics of immersed rigid-body can be simulated. The torque due to drag and lift forces is $\tau_t = \vec{p} \times f_t$ where $\vec{p}$ is the vector from the center of mass to the center of pressure of the body.

2.2 Stochastic Model

A stochastic model for the motion of suspended fluid particles was proposed by Langevin. According to this model, the velocity increments of particles in continuous time steps are highly correlated called an Ornstein-Uhlenbeck process [Uhlenbeck and Ornstein 1930]. The model can be applied to describe the Brownian motion of objects undergoing vortical loads from the surrounding turbulent flow [Makino and Doi 2004; Yuan et al. 2011].

For a statistically isotopic turbulence, a first-order stochastic differential equation can be adopted to represent the velocity increments:

$$dX(t) = D_1 X(t)dt + \sqrt{D_2}dW \tag{6}$$

This equation is also known as Langevin equation, where $X(t)$ is the velocity of the fluid particle; D_1 and D_2 are the drift and diffusion functions, which reveal the properties of the turbulent flow; and W is a continuous-time stochastic process of Brownian motion called Wiener process. In our implementations, the Wiener process is calculated by a normal distribution with mean of zero and variance of the time interval Δt.

D_1 and D_2 are the functions of the turbulent kinetic energy and its dissipation rate in a turbulent flow [Pope 1983]. They are defined by the following formulations:

$$D_1 = -(\frac{1}{2} + \frac{3}{4}C_k)\frac{\varepsilon}{k}, \qquad D_2 = C_k\varepsilon \tag{7}$$

where k and ε are kinetic energy and its dissipation rate of the surrounding turbulent flow; $C_k = 2.1$ is a Kolmogorov coefficient.

The Langevin equation can also be used to represent the velocity increments of a rigid body with arbitrary shape moving in a turbulent flow, where the drift function in Equation 6 is omitted for rotational velocity increments of the body as a rotational Brownian motion [Makino and Doi 2004].

2.3 Time Integration

We propose a fractional-step method for solving the velocity $u = (\omega, v) \in R^6$ from Equation 2 which includes three steps as follows:

$$u \xrightarrow{(f_g,\tau_g)} \dot{u} \xrightarrow{Langevin} \ddot{u} \xrightarrow{(f_t,\tau_t)} u^{new}$$

The first step is to numerically solve for a guess velocity $\dot{u}$ while not considering the vortical loads in Equation 2. Then, We utilize the Langevin equations of both translational and rotational velocities to obtain an intermediate velocity $\ddot{u}$.

$$\ddot{u} = \dot{u} - \begin{pmatrix} -\sqrt{C_k\varepsilon\Delta t}\vec{\xi_1} \\ \chi(\frac{1}{2} + \frac{3}{4}C_k)\dot{v}\Delta t - \sqrt{C_k\varepsilon\Delta t}\vec{\xi_2} \end{pmatrix} \tag{8}$$

Where $\chi = \varepsilon/k$, $\vec{\xi_1}$ and $\vec{\xi_2}$ are the vectors of normal Gaussian distributed variables $Norm(0,1)$ with mean zero and unit deviation. The vectors are generated using the Box-Muller algorithm [Press et al. 1992].

After calculating the vortical loads (f_t, τ_t) by Equation 5 using $U = \ddot{v} \in \ddot{u}$, the last step is to solve Equation 2 to obtain u^{new} for next time step. The parameters (χ, ε) in Equation 8 represent the characteristics of the surrounding turbulent flow, which are computed by two-equation k-ε model in a precomputation step.

3 Turbulence Model

The fluid velocity u of a turbulent flow is decomposed into a mean flow $\langle u \rangle$ and a fluctuating velocity u' ($u = \langle u \rangle + u'$) by Reynolds decomposition. A k-ε turbulent model is adopted in this work. This model is a semi-empirical model based on the transport equations, which consists of two coupled equations for the turbulent kinetic energy k and its dissipation rate ε. The energy transport equations are defined as follows [Pfaff et al. 2010]:

$$D_t k = G - \varepsilon \tag{9}$$

$$D_t \varepsilon = \chi(C_1 G - C_2 \varepsilon) \tag{10}$$

where C_1 and C_2 are empirical constants with values $C_1 = 1.44$ and $C_2 = 1.92$ [Launder and Sharma 1974]. G represents the generation of turbulent kinetic energy due to the mean velocity gradients and can be defined in terms of the strain tensor $S_{ij} = \frac{1}{2}(\frac{\partial\langle u\rangle_i}{\partial x_j} + \frac{\partial\langle u\rangle_j}{\partial x_i})$ of the flow:

$$G = 2v_T \sum_{ij} S_{ij}^2 \tag{11}$$

where v_T is the turbulent viscosity describing the small-scale turbulent motion as a viscous diffusion scale in the turbulent model.

Algorithm 1 Pseudo-code for our stochastic model.

Precompute Kirchhoff tensor
// **Precompute turbulent parameters**
Boundary conditions ←Eq.(13)
while not stopped **do**
    // solve the mean flow $\langle u \rangle$
    Convection by semi-Langrangian
    Pressure projection by Poisson solver
    // Energy transport
    Compute turbulent viscosity v_T ←Eq.(12)
    Compute strain tensor term G ←Eq.(11)
    Integrate turbulent parameters (k, ε) ←Eqs.(9)(10)
    $t = t + \Delta t$
end while
Save (χ, ε)
// **Runtime process**
Given initial conditions of the rigid body
Timestep $t = 0$
while not arrive ground **do**
    Calculate the guess velocity $\dot{u}$ ←Eqs.(2)(3)(4)
    Query χ_t and ε_t
    Calculate intermediate velocity $\ddot{u}$ ←Eq.(8)
    Calculate vortical loads (f_t, τ_t) ←Eq.(5)
    Solve the GKE equations ←Eq.(2)
    Integrate (R, x) ←Eq.(1)
    $t = t + \Delta t$
end while

Turbulent viscosity v_T is defined as:

$$v_T = C_\mu \frac{k^2}{\varepsilon} \tag{12}$$

where $C_\mu = 0.09$ is an empirical constant. In cases of turbulent flows with high Reynolds numbers, the initial state (k_0, ε_0) is given as follows in terms of the terminal falling velocity U_0.

$$k_0 = \frac{3}{2}U_0^2; \qquad \varepsilon_0 = C_\mu^{\frac{3}{4}} k_0^{\frac{3}{2}} / \Delta l \tag{13}$$

where Δl is the length scale of MAC grid cell in mean flow simulation, and $U_0 = \sqrt{(\rho_s/\rho_f - 1)gt}$ is terminal falling velocity of the rigid body depending on the thickness t of the body.

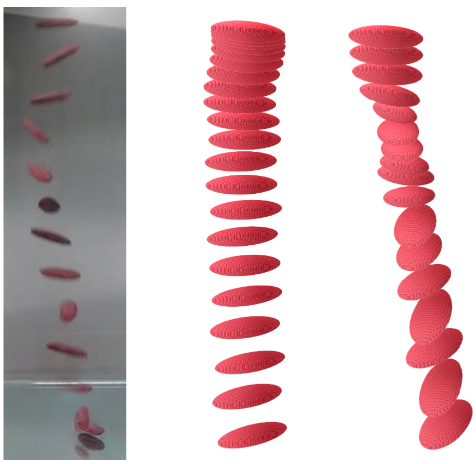
The turbulent parameters (χ, ε) are explicitly solved with finite difference scheme from Equation 9, where we apply a standard fluid solver to obtain the mean velocities $\langle u \rangle$ of the mean flow simulation with an inflow fluid velocity U_0.

4 Results

Our simulations were implemented on a standard PC with an Intel Core i7 CPU with 3.20 GHz and 12.0 GB RAM. The implementation of turbulent model utilized a $32 \times 32 \times 8$ typical MAC staggered grid with semi-Langragian convection to obtain the base mean flow with a computation cost 182 ms. The precomputation of Kirchhoff tensor is similar to the work [Weissmann and Pinkall 2012] and takes 53 ms for a model with 1280 triangles. We use a geometric Lie group integrator to solve rigid body dynamics in runtime as described in [Kobilarov et al. 2009]. The average computation cost is around 20 ms for one time step. The whole simulation flow is described in Algorithm 1.

Figure 1 shows a comparison between our simulation result and a video of a gliding paper airplane. The paper airplane is made by a $8.3 \times 8.3 \times 0.01$ (cm) print paper. The model of the paper airplane used in the simulation has a closed shape with 288 triangles, and the initial velocity is set to be 20 cm/s in horizontal direction. The simulation result exhibits two turning motions (turning front and turning sideways) which are caused by the viscous effect of the surrounding airflow. The turning motions are coincident with the observation from the reference video.

Figure 3 shows the discrete frames from ground truth, and simulation results of previous work and our approach for a rubber ellipsoid falling in water. The rubber ellipsoid model has 1280 triangles with semi-principal axes of length 1 cm, 2 cm and 4 cm. A small-scale oscillation can be found in the horizontal direction (Figure 3 (b)) by the approach of previous work [Weissmann and Pinkall 2012]. In contrast to previous work, the vortical loads due to the viscous effect of the surrounding turbulent flow can be indicated properly using our approach. Since the oscillations of the rigid body in different directions (Figure 3 (a)) captured in the ground truth are also found in our simulation result (Figure 3 (c)), our stochastic model is more realistic than previous work.



(a) (b) (c)

Figure 3: *Comparison among (a) ground truth,(b) previous work and (c) our approach. Ground truth shows oscillations generated in different directions (the shorter silhouettes manifest the oscillation in three dimensions.), previous work takes account only Kirchhoff tensor, whereas our approach has concerned the vortical loads.*

5 Conclusion

We presented a stochastic model for realistic simulations of rigid bodies in viscous flows, evaluating the vortical loads due to the surrounding turbulent flow in immersed rigid-body dynamics by Langevin equations after a preprocess utilizing turbulent energy model. We proposed a fractional-step method to solve the rigid-body dynamics in runtime process and our approach allows real-time simulations for interactive applications. However, some limitations also exist in our model: 1) the simulation results are sensitive to the initial conditions; 2) our model is not able to analysis motion patterns of immersed rigid-bodies quantitatively. We would like to combine our model with a statistical model of motion patterns [Xie and Miyata 2013] to solve these problems in the future.

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