I211 Mathematical Logic

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Q 1. Using the order relation $<$, equality $=$, the function $+$ and the constants $0, 1$, write sentences (of predicate logic) to describe the following statements .

1. 0 is the minimal element.

2. There is no maximal element.

3. If $x < x + y$, then y is not 0.

4. If there is no x such that $x + x = y$, then there is z such that $(z + z) + 1 = y$.

Q 2. Let $\mathcal{L} = (\times; \leq)$. Answer the truth (true or false) of the following formulas in $(\mathbb{N}; \times; \leq)$, $(\mathbb{Z}; \times; \leq)$ or $(\mathbb{Q}; \times; \leq)$. (Here, $\times$ is a binary function symbol, $\leq$ is a binary relation symbol and they are understood as the usual product and the order "not greater than".)

1. $\forall x \forall y (x \leq y \vee y \leq x)$.

2. $\exists z \forall x (z \leq x)$.

3. $\forall x \forall y ((x \leq y \wedge \neg x = y) \to \exists z (x \leq z \wedge z \leq y \wedge \neg x = z \wedge \neg y = z))$.

4. $\forall x \forall y \forall z (x \leq y \to x \times z \leq y \times z)$.

Q 3. Let $\mathcal{L} = (R(x); f(x); c)$. Enumerate all subformulas of the following, and find free variables (FV) and bounded variables (BV) of each of them.

$$\forall x (f(c) = x \to ((x = y \to f(c) = y) \wedge (\forall z (y = c \vee y = f(x))))).$$

Q 4 (review)**.** Show that the following are provable in LK.

1. $(A \to B) \to (\neg B \to \neg A)$.
2. $(A \to B) \lor (B \to A)$.
3. $((A \to B) \land (B \to C)) \to (A \to C)$.
4. $(\neg A \land A) \to B$.

Q 5. Show that the following are provable in LK.

1. $\forall x(\varphi \land \psi) \to \forall x\psi$
2. $\exists x\psi \to \exists x(\varphi \lor \psi)$
3. $(\exists x\varphi \to \forall x\psi) \to \forall x(\varphi \to \psi)$
4. $\neg\exists x\neg\varphi \to \forall x\varphi$

Q 6. Let $\mathcal{L} = (c; f(\cdot), g(\cdot); R(\cdot), S(\cdot))$. Show that the following are provable in LK.

1. $\forall x(R(x) \to S(x)) \to (\exists y R(y) \to \exists y S(y))$
2. $\neg(f(c) = c) \to \exists x\exists y\neg(x = y)$
3. $\forall x(g(f(x)) = x) \to \forall x\forall y(f(x) = f(y) \to x = y)$
4. $\forall x(f(g(x)) = x) \to \forall x\exists y(x = f(y))$

(advanced): Check the meaning of the above 3 and 4, and consider the reason why they are always true.

Q 7. Show that the following are valid.

1. $\forall x((A \land \neg A) \to B)$
2. $(\exists x A) \lor (\forall x\neg A)$
3. $\neg(\exists x(A \lor B)) \to \neg(\exists x A \lor \exists x B)$
4. $(\forall x A \to \exists y B) \to (\forall y\neg B \to \exists x\neg A)$

Q 8. Let $\mathcal{L} = (c; f(\cdot), g(\cdot); R(\cdot), S(\cdot))$. Show that the following are valid.

1. $\forall x(x = c \to R(f(x))) \to R(f(c))$
2. $\forall x(R(x) \to S(f(x))) \to (\exists x R(x) \to \exists y S(y))$
3. $\forall x \forall y(f(x) = g(y)) \to \forall x(f(x) = g(c))$
4. $(\exists x S(x) \land \neg S(c)) \to \exists x \exists y(\neg x = y)$