I211 Mathematical Logic

横山啓太

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Q 9. Let $\mathcal{L} = (c; f(\cdot); S(\cdot))$. Answer whether the following $\mathcal{L}$-formulas are satisfiable or not, and explain the reason.

1. $\forall x(S(x) \leftrightarrow \neg S(f(x))) \land S(c)$.

2. $\forall x \forall y(f(x) = f(y) \to x = y) \land \forall x((x = c) \leftrightarrow \neg S(x)) \land \forall x(S(x) \to \exists y(x = f(y)))$.

3. $\exists x S(x) \land \exists x \forall z(S(z) \to z = x) \land \forall x(S(x) \to S(f(x))) \land \forall x(\neg x = f(x))$.

Q 10. Let $\mathcal{L}$ be a language, $\mathcal{M}$ be an $\mathcal{L}$-structure, and φ be an $\mathcal{L}$-formula with $FV(\varphi) = \{x\}$. Show (by the definition of the truth) that the following 1 and 2 have the same truth value in $\mathcal{M}$.

1. $\forall x(\neg\varphi)$.

2. $\neg(\exists x \varphi)$.

Q 11. Let $\mathcal{L} = \{e; \cdot\}$, and let G be the theory of groups, *i.e.*, G consists of the (universal closure of the) following sentences.

1. $(x \cdot y) \cdot z = x \cdot (y \cdot z)$

2. $x \cdot e = x \land e \cdot x = x$

3. $\exists y(y \cdot x = e \land x \cdot y = e)$

Show the following.

1. $G \models \exists y \forall x(x \cdot y = x)$.

2. $G \not\models \forall x \forall y (x \cdot y = y \cdot x)$.

Q 12. Let $\mathcal{L} = \{e; \cdot\}$, and let G be the theory of groups. Show the following.

1. $G \vdash \forall x \forall y \forall z ((y \cdot x = e \wedge z \cdot x = e) \rightarrow y = z)$
2. $G \vdash \forall x \forall y (y \cdot x = e \rightarrow x \cdot y = e)$

Q 13 (advanced)**.** Let $\mathcal{L} = (0, 1; +, \times)$. Robinson arithmetic Q consists of the following:

1. $\neg(x + 1 = 0)$
2. $(x + 1 = y + 1) \rightarrow (x = y)$
3. $(y = 0) \vee \exists x (x + 1 = y)$
4. $x + 0 = x$
5. $x + (y + 1) = (x + y) + 1$
6. $x \times 0 = 0$
7. $x \times (y + 1) = (x \times y) + x$

Peano arithmetic PA is defined as

$$\mathrm{PA} = \mathrm{Q} \cup \{(\varphi(0) \wedge \forall x (\varphi(x) \rightarrow \varphi(x+1))) \rightarrow \forall x \varphi(x) : \varphi \text{ is an } \mathcal{L}\text{-formula}\}.$$

We use n for the the abbreviation of the term $((\ldots(\overbrace{1+1)+1)\cdots+1}^{n})$. Prove the following.

1. $\mathrm{Q} \vdash 2 + 3 = 5$
2. $\mathrm{Q} \vdash x \times 2 = x + x$
3. $\mathrm{PA} \vdash \forall x \forall y (x + y = y + x)$