I211 Mathematical Logic

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Q 14. Let $\mathcal{L} = \{c; f(\cdot)\}$. Answer whether the following are provable, disprovable or unprovable but not disprovable, and explain the reason.

1. $\forall x \forall y (f(x) = f(y) \to x = y) \land \forall x (\neg f(x) = c)$.
2. $\exists x \exists y \exists z (\neg x = y \land f(z) = x \land f(z) = y)$.
3. $\forall x \forall y \forall z (x = y \lor y = z \lor x = z) \to (\exists x \exists y f(x) \neq f(y) \to \forall x \exists y f(y) = x)$.

Q 15. Let $\mathcal{L}$ be a language, Γ be a finite $\mathcal{L}$-theory and φ be an $\mathcal{L}$-sentence. Show the following.

Γ, φ is inconsistent if and only if $\Gamma \vdash \neg\varphi$.

Q 16 (advanced)**.** For the soundness theorem, we need to check the following by induction on the height of LK-proofs.

($*$) Assume $FV(\Gamma) \cup FV(\Delta) = \vec{x} = (x_1, \dots, x_k)$. Then, for any $\mathcal{L}$-structure $\mathcal{M} = (M; \dots)$ and for any $\vec{a} = (a_1, \dots, a_k) \in \bar{M}$

$$\mathcal{M} \models \bigwedge \Gamma[\vec{a}/\vec{x}] \Longrightarrow \mathcal{M} \models \bigvee \Delta[\vec{a}/\vec{x}]$$

In other words, we need to show

(i) ($*$) holds for initial sequents, and,

(ii) for each inference rule, if ($*$) holds for the upper sequent, then ($*$) holds for the lower sequent.

Answer the following.

1. Initial sequents for equality satisfy the condition $(*)$.

2. Check that cut, $\neg$R, $\wedge$R, $\vee$L rules satisfy (ii).

3. Check that $\exists$R, $\exists$L rules satisfy (ii).

Q 17 (advanced)**.** Let $\mathcal{L} = \{R(x), S(x); a\}$. Show that the following are provable in LK.

1. $\exists x(R(x) \to \forall y R(y))$.

2. $\exists x((R(a) \to \exists y S(y)) \to (\forall z(R(z) \to \neg S(z)) \to R(x)))$.