A CDCL-style Calculus for Solving Non-linear Constraints [1]

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Main Problem

Checking satisfiability of (non-)linear constraints.

Linear:

$$\begin{aligned} 5x - 10y - 2z &\leq 1/2 \\ 6x - 1/3y + 2z &> 0 \\ -10x + 5y - 31z &\geq 0 \end{aligned}$$

Main Problem

Checking satisfiability of (non-)linear constraints.

Polynomial:

$$
\begin{aligned}
3x^2y - 10yzx - 2z &\leq 5 \\
-5yz^2 - 1/3y + 2z &> 1 \\
x + 5xy - 11xz &\geq 7
\end{aligned}
$$

Main Problem

Checking satisfiability of (non-)linear constraints.

Non-linear with transcendental functions

$$2\sin^2 x - 5\cos y^2 - 2z \leq 1/2 \quad \vee \quad e^{x^{-2}} + zy < y$$
$$4x - 1/3y + 2zx > 0$$
$$x^2 - y^2 - z \geq 0$$

Motivation:

- Verification: of hybrid; embedded systems; programs etc.
- Proof assistance for mathematics which rely on computations with non-linear constraints such as Hales proof of Kepler's conjecture.

In most cases the problem of solving non-linear constraints is undecidable or relates to open problems in maths.

Overview of our approach

1. separated linear form $\mathcal{L} \wedge \mathcal{N}$
2. ksmt calculus – conflict-driven calculus for solving non-linear constraints
3. local linearisations for resolving non-linear conflicts
    - approximation of non-linear problem by incremental linearisations
    - related work [A. Cimatti, A. Griggio, A. Irfan, M. Roveri, and R. Sebastiani'18;...]
4. δ-complete decision procedure for bounded instances

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4. δ-complete decision procedure for bounded instances

New class $\mathcal{F}_{\mathrm{DA}}$ – functions with decidable rational approximations

- Checking "non-linear conflicts" is decidable for functions in $\mathcal{F}_{\mathrm{DA}}$
- inspired by computable analysis

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New class $\mathcal{F}_{\mathrm{DA}}$ – functions with decidable rational approximations

- Checking "non-linear conflicts" is decidable for functions in $\mathcal{F}_{\mathrm{DA}}$
- inspired by computable analysis
- $\mathcal{F}_{\mathrm{DA}}$ includes:
  - Multivariate polynomials x^2yz^5
  - Transcendental functions: $\exp, \ln, \log_b, \sin, \cos, \tan, \arctan \ldots$
  - Discontinuous functions: step-functions; piecewise linear/polynomial functions

From Logic to Arithmetic: The linear case

From Logic to Linear Arithmetic: Resolution

Motivation:
How to extend efficient SAT technology to other domains/theories?

- Black-box: CDCL(T) – separate Boolean structure and theory
- SAT-encodings: bit-vectors etc.
- White-box: extend SAT calculi to other domains

From Logic to Linear Arithmetic: Resolution

propositional	linear arithmetic
clauses	linear inequalities
$\neg x_1 \vee x_2 \vee \cdots \vee x_n$	$-5x_1 + 3x_2 + \cdots + 0.5x_n + 17 \geq 0$
clause resolution	inequality resolution
$\dfrac{\neg x \vee C \quad x \vee D}{C \vee D}$	$\dfrac{-ax + p \geq 0 \quad bx + q \geq 0}{bp + aq \geq 0}$

Combine model search and proof search

Conflict resolution – combination of model search and proof search

- Iteratively assign values (A) to variables $x_1 \mapsto 0 :: x_2 \mapsto 0.2 :: \ldots :: x_n \mapsto 5$
- If all constraints evaluate to true then – done
- Otherwise, we have a conflict
  1. resolve (R)
  2. backjump (B)
  3. refine assignment (A)

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- Conflict Resolution [Korovin, Tsiskaridze, Voronkov, 2009]
- GDPLL [McMillan, Kuehlmann, Sagiv 2009]
- bound propagation [Korovin, Voronkov, 2011]
- MCSAT/NLSAT [Jovanović, de Moura, 2012/2013]
- CDSAT [Bonacina, Graham-Lengran, Shankar, 2017]
- ...

Example

$$
\begin{array}{rrrrrrrrrrrl}
x_4 & - & 2x_3 & & & + & x_1 & + & 5 & \geq & 0 & (1)\\
x_4 & + & 2x_3 & + & x_2 & & & + & 3 & \geq & 0 & (2)\\
-x_4 & - & x_3 & - & 3x_2 & - & 3x_1 & + & 1 & \geq & 0 & (3)\\
-x_4 & + & 2x_3 & + & 2x_2 & + & x_1 & + & 6 & \geq & 0 & (4)\\
 & & x_3 & & & + & 3x_1 & - & 1 & \geq & 0 & (5)\\
 & & -x_3 & + & x_2 & - & 2x_1 & + & 5 & \geq & 0 & (6)
\end{array}
$$

variable				
bounds				
assignment				

Example

$$
\begin{array}{rcrcrcrcrcrr}
x_4 & - & 2x_3 & & & + & x_1 & + & 5 & \geq & 0 & (1) \\
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variable	x_1			
bounds				
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Example

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variable	x_1			
bounds	$(-\infty, \infty)$			
assignment				

Example

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variable	x_1			
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 & & x_3 & & & + & 3x_1 & - & 1 & \geq & 0 \quad {\color{red}(5)}\\
 & & -x_3 & + & x_2 & - & 2x_1 & + & 5 & \geq & 0 \quad {\color{blue}(6)}
\end{array}
$$

variable	x_1	x_2	x_3
bounds	$(-\infty, \infty)$	$(-\infty, \infty)$	$[{\color{red}1}; {\color{blue}5}]$
assignment	$x_1 \mapsto 0$	$x_2 \mapsto 0$	

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assignment	$x_1 \mapsto 0$	$x_2 \mapsto 0$	$x_3 \mapsto 4$

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bounds	$(-\infty, \infty)$	$(-\infty, \infty)$	$[1; 5]$	$[3; -3]$
assignment	$x_1 \mapsto 0$	$x_2 \mapsto 0$	$x_3 \mapsto 4$	

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assignment	$x_1 \mapsto 0$	$x_2 \mapsto 0$	$x_3 \mapsto 4$	

Conflict: $\text{res}_{x_4}((1), (3)) \rightsquigarrow (7)$

Example

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\end{array}
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bounds	$(-\infty, \infty)$	$(-\infty, \infty)$	$[1; 5]$	$[3; -3]$
assignment	$x_1 \mapsto 0$	$x_2 \mapsto 0$	$x_3 \mapsto 4$	

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variable	x_1	x_2	x_3
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variable	x_1	x_2	x_3
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\end{array}
$$

variable	x_1	x_2	x_3
bounds	$(-\infty, \infty)$	$(-\infty, \infty)$	$[1; 2]$
assignment	$x_1 \mapsto 0$	$x_2 \mapsto 0$	$x_3 \mapsto 1$

Example

$$
\begin{array}{rcrcrcrcrcl}
x_4 & - & 2x_3 & & & + & x_1 & + & 5 & \geq & 0 \quad (1)\\
x_4 & + & 2x_3 & + & x_2 & & & + & 3 & \geq & 0 \quad (2)\\
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variable	x_1	x_2	x_3	x_4
bounds	$(-\infty, \infty)$	$(-\infty, \infty)$	$[1; 2]$	
assignment	$x_1 \mapsto 0$	$x_2 \mapsto 0$	$x_3 \mapsto 1$	

Example

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variable	x_1	x_2	x_3	x_4
bounds	$(-\infty, \infty)$	$(-\infty, \infty)$	$[1; 2]$	$[-3; 0]$
assignment	$x_1 \mapsto 0$	$x_2 \mapsto 0$	$x_3 \mapsto 1$	

Example

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variable	x_1	x_2	x_3	x_4
bounds	$(-\infty, \infty)$	$(-\infty, \infty)$	$[1; 2]$	$[-3; 0]$
assignment	$x_1 \mapsto 0$	$x_2 \mapsto 0$	$x_3 \mapsto 1$	$x_4 \mapsto -1$

SAT

Example

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x_4 & + & 2x_3 & + & x_2 & & & + & 3 & \geq & 0 \quad (2)\\
-x_4 & - & x_3 & - & 3x_2 & - & 3x_1 & + & 1 & \geq & 0 \quad (3)\\
-x_4 & + & 2x_3 & + & 2x_2 & + & x_1 & + & 6 & \geq & 0 \quad (4)\\
 & & x_3 & & & + & 3x_1 & - & 1 & \geq & 0 \quad (5)\\
 & & -x_3 & + & x_2 & - & 2x_1 & + & 5 & \geq & 0 \quad (6)\\
 & & -x_3 & - & x_2 & - & \frac{2}{3}x_1 & + & 2 & \geq & 0 \quad (7)
\end{array}
$$

variable	x_1	x_2	x_3	x_4
bounds	$(-\infty, \infty)$	$(-\infty, \infty)$	$[1; 2]$	$[-3; 0]$
assignment	$x_1 \mapsto 0$	$x_2 \mapsto 0$	$x_3 \mapsto 1$	$x_4 \mapsto -1$

SAT

CR is correct and terminating. [Korovin, Tsiskaridze, Voronkov; CP'09]

Fourier-Motzkin vs Conflict Resolution

Example:

$$
\begin{array}{rcrcrcrcrcrcl}
2x_5 & - & 3x_4 & + & x_3 & - & 3x_2 & - & 2x_1 & + & 3 & \geq & 0 \\
2x_5 & + & x_4 & - & 2x_3 & & & - & 2x_1 & + & 2 & \geq & 0 \\
-x_5 & & & & & + & 3x_2 & + & x_1 & + & 2 & \geq & 0 \\
-3x_5 & & & + & 2x_3 & & & - & 3x_1 & - & 2 & \geq & 0 \\
x_5 & - & 2x_4 & & & - & 2x_2 & + & 3x_1 & - & 2 & \geq & 0 \\
-2x_5 & + & 2x_4 & - & 3x_3 & - & x_2 & + & 2x_1 & + & 3 & > & 0 \\
3x_5 & - & 2x_4 & + & 2x_3 & + & 3x_2 & + & 2x_1 & + & 1 & > & 0 \\
x_5 & & & & & & & + & 2x_1 & + & 2 & > & 0 \\
 & & 2x_4 & - & x_3 & - & 3x_2 & - & x_1 & + & 3 & = & 0
\end{array}
$$

Fourier-Motzkin: Generates over 280 million linear inequalities.

Conflict Resolution: Generates 21 linear inequalities.

ksmt calculus – extending conflict resolution to non-linear constraints

Existentially quantified formula in CNF over predicates over $(\mathbb{R}, \mathcal{F}_{\mathsf{lin}} \cup \mathcal{F}_{\mathsf{nl}}, <, \leq, >, \geq)$.

Example: $$\exists x, y : \Big(((\sin x)^2 + (\cos x)^2 < 1) \vee (\exp x < y) \Big) \wedge \Big(4 \cdot x > y \Big)$$

An assignment $\alpha : V \to \mathbb{Q}$ is a solution to such a CNF $\mathcal{C}$ over variables V iff

- α assigns all quantified variables
- for each clause $C \in \mathcal{C}$ there is $\ell \in C$ with evaluates to true, in symbols: $[\![\ell]\!]^\alpha = \mathsf{true}$

Problem: finding solution to $\mathcal{C}$ or showing that none exists.

Overview of the ksmt approach

Main ingredients of our approach:

1. separated linear form $\mathcal{L} \wedge \mathcal{N}$
2. ksmt calculus – conflict-driven calculus for solving non-linear constraints
3. local linearisations for resolving non-linear conflicts

Separated Linear Form

Separated linear form: $\mathcal{L} \wedge \mathcal{N}$

- $\mathcal{L}$ – linear inequalities: $q_1 x_1 + q_2 x_2 + \dots + q_n x_n + q_0 \diamond 0$

$$\begin{aligned} 2x - 4y - 2u - 2 &> 0 \\ -x + 2y + 3u + 1 &> 0 \\ 4y + 2u + 1 &\geq 0 \end{aligned}$$

- $\mathcal{N}$ – non-linear units: $x \diamond f(\bar{t})$

$$\begin{aligned} y &> \sin(x^2) \\ u &\leq y^2 x \\ x &\geq \mathrm{e}^{-u} \end{aligned}$$

Separated Linear Form

Transforming non-linear constraints into separated linear form.

- Monotonic flattening: introduce fresh variables for non-linear terms.

$$5\sin(x^2) - 3xy + 2x - 13 \geq 0 \mapsto$$

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$$x_2 \geq xy$$

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$$5x_1 - 3x_2 + 2x - 13 \geq 0$$
$$x_1 \leq \sin(x^2)$$
$$x_2 \geq xy$$

Lemma. Any CNF over non-linear constraints can be efficiently transformed into separated linear form using monotonic flattening.

ksmt calculus

- Transition rules define relation $\Rightarrow$ on states $(\alpha, \mathcal{L}, \mathcal{N})$.
  - (partial) assignment α
  - linear inequalities $\mathcal{L}$
  - non-linear units $\mathcal{N}$
- initial state is $(\text{nil}, \mathcal{L}_0, \mathcal{N}_0)$ for formula in separated linear form $\mathcal{L}_0 \wedge \mathcal{N}_0$.
- sat and $unsat$ are final states.

$\mathcal{N}_0$ remains unchanged under $\Rightarrow$ transformations.

ksmt calculus

Rules	$(\alpha, \mathcal{L}, \mathcal{N}) \Rightarrow \circ$	

ksmt calculus

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(B) backjump	$(\gamma, \mathcal{L}, \mathcal{N})$	to γ maximal conflict-free prefix of α
(L) linearise	$(\alpha, \mathcal{L} \cup L, \mathcal{N})$	L linearisation, excluding the non-linear conflict
(F^{sat})	sat	all variables are assigned, no linear conflict and (A), (R), (B), (L) are not applicable
(F^{unsat})	$unsat$	$[\![\mathcal{L}]\!]^{nil} =$ false

$[\![\cdot]\!]^{\alpha}$ is the partial evaluation under α.

Local linearisations

Definition

Assignment α with non-linear conflict $[\![P]\!]^\alpha = \text{false}$ for a non-linear unit P. A linear clause L is a linearisation if

- for any assignment β, $[\![P]\!]^\beta = \text{true}$ implies $[\![L]\!]^\beta = \text{true}$, and
- $[\![L]\!]^\alpha = \text{false}$.

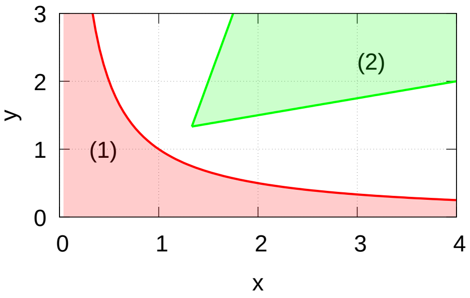
unsat example run using Interval linearisation

$$\mathcal{C} = \underbrace{(y \le 1/x)}_{P} \wedge (x/4 + 1 \le y) \wedge (y \le 4 \cdot (x-1))$$

Linearisation of conflicts (x, y) at α here:

- choose $d := (1/[\![x]\!]^\alpha + [\![y]\!]^\alpha)/2$,
- $L = \left(x \le 1/d \ \vee \ y \le d\right)$

rule	α	note



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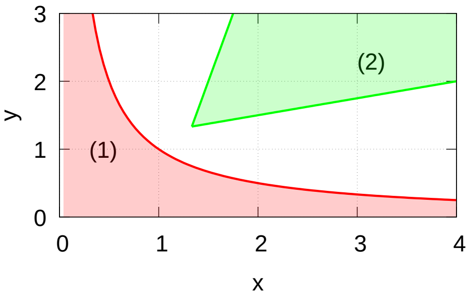
rule	α	note
(A)	$x \mapsto 2$	



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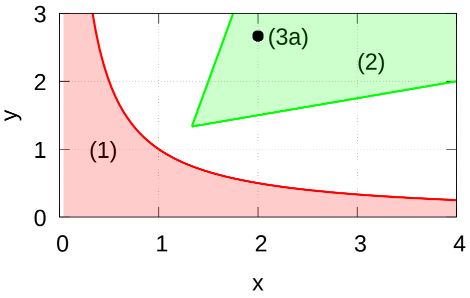

unsat example run using Interval linearisation

$$\mathcal{C} = \underbrace{(y \le 1/x)}_{P} \land (x/4 + 1 \le y) \land (y \le 4 \cdot (x-1)) \land \left((x \le \tfrac{12}{19}) \lor (y \le \tfrac{19}{12})\right)$$

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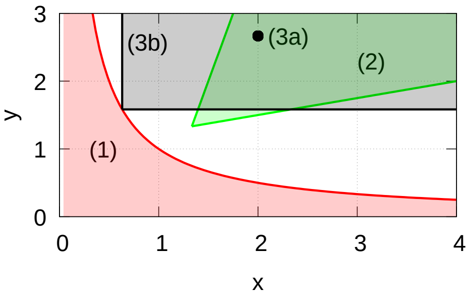

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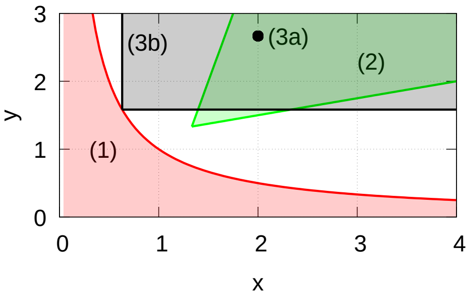

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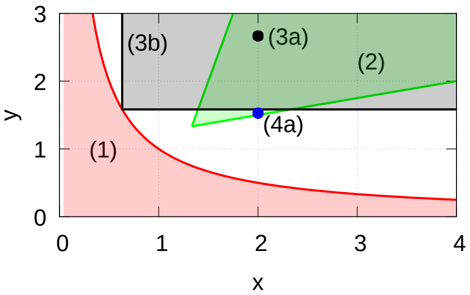

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(B)	$x \mapsto 2$	
(A)	$x \mapsto 2,\ y \mapsto \frac{84}{55}$	(4a)

unsat example run using Interval linearisation

$$\mathcal{C} = \underbrace{(y \leq 1/x)}_{P} \wedge (x/4 + 1 \leq y) \wedge (y \leq 4 \cdot (x-1))$$
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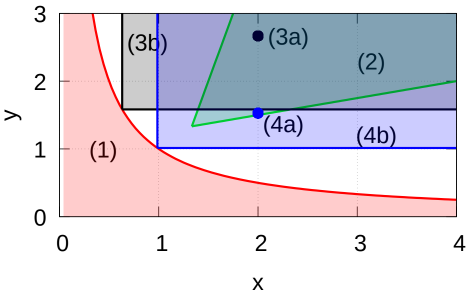

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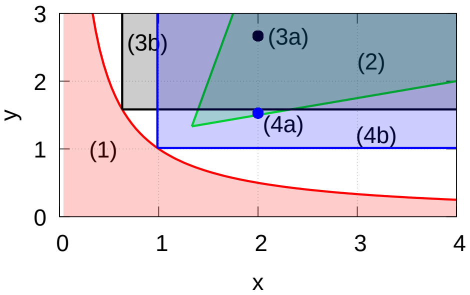

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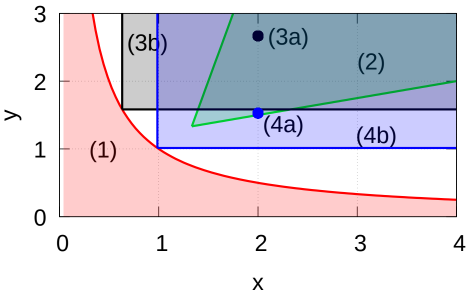

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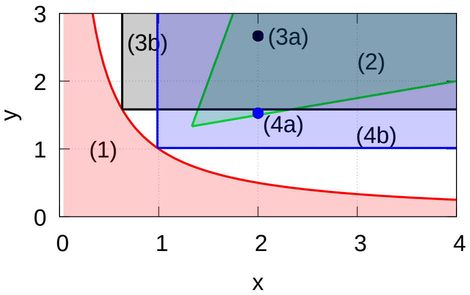

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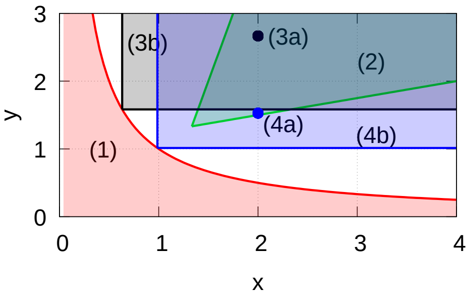

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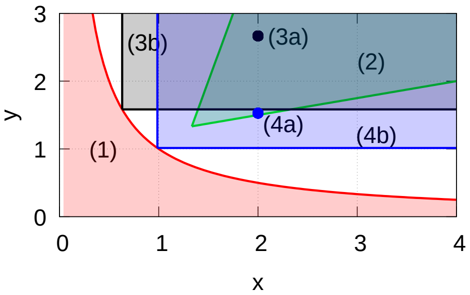

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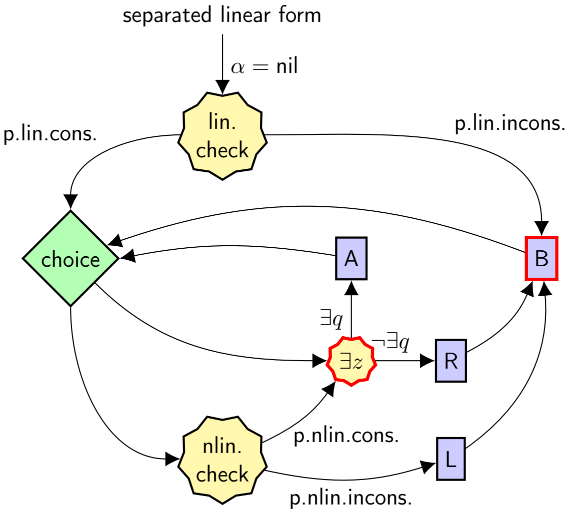
(R)	$x \mapsto 2$	on y
(B)		
(R)		on x
n/a		unsat

Core of ksmt calculus



Some properties of the ksmt calculus

Lemma

Let $\mathcal{C}$ be a formula in separated linear form, let $S_i = (\alpha_i, \mathcal{L}_i, \mathcal{N})$ be states with $S_0 \Rightarrow S_1 \Rightarrow \cdots \Rightarrow S_n$ and S_0 the initial state. Then

- For any total assignment $\beta : V \to \mathbb{Q}$: $[\![\mathcal{L}_i \cap \mathcal{N}]\!]^\beta = [\![\mathcal{L}_{i+1} \wedge \mathcal{N}]\!]^\beta$.
- If no rule is applicable to S_n, then S_n is conflict-free iff $\mathcal{C}$ has a solution.

Corollary (Soundness)

Let $S = (\alpha, \mathcal{L}, \mathcal{N})$ be derivable from $(\text{nil}, \mathcal{L}_0, \mathcal{N})$ in ksmt.

- If (F^{sat}) is applicable to S, then α is a solution to $\mathcal{L}_0 \wedge \mathcal{N}$.
- If (F^{unsat}) is applicable to S, then $\mathcal{L}_0 \wedge \mathcal{N}$ is unsatisfiable.

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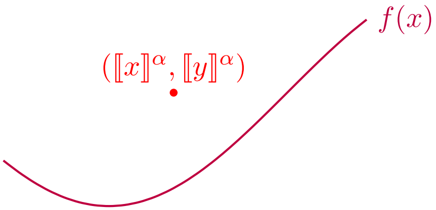
Lemma (Progress)

After at most $(\#\text{variables} + 2)$ steps the search space is reduced.

Deciding non-linear conflicts

$f(x) \geq y$, $\alpha : V \to \mathbb{Q}$



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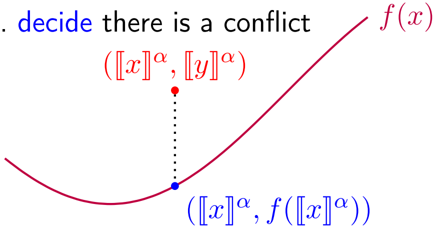
1. decide there is a conflict



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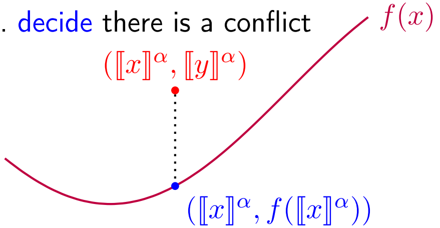


Computable Analysis: theory of computations on continuous structures: $\mathbb{R}$, $C([0,1],\mathbb{R})$, ...

- efficient implementation: iRRAM [Müller '00]

Definition (Cauchy representation of $\mathbb{R}$)

$x \in \mathbb{R}$ is computable iff $\tilde{x} : \mathbb{N} \to \mathbb{Q}$ is computable with $\forall n : |\tilde{x}(n) - x| \leq 2^{-n}$.

In general, $f(\llbracket x \rrbracket^\alpha) \geq \llbracket y \rrbracket^\alpha$ is not decidable, so we need more information about f.

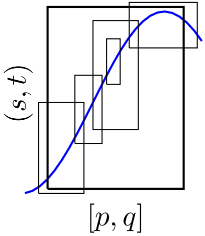
Approximability

Definition

A partial function $f : \mathbb{R} \to \mathbb{R}$ is called approximable iff

$$\{(p, q, s, t) : f([p, q]) \subset (s, t)\} \subset \mathbb{Q}^4$$

is computably enumerable.



Lemma

For total continuous real functions, approximability coincides with the notion of computability known from Computable Analysis.

The class $\mathcal{F}_{\mathrm{DA}}$

Definition

$\mathcal{F}_{\mathrm{DA}}$ – functions with decidable rational approximations; $g \in \mathcal{F}_{\mathrm{DA}}$ if

- $\operatorname{dom} g \cap \mathbb{Q}^n$ decidable,
- $\operatorname{graph} g \cap \mathbb{Q}^n \times \mathbb{Q}$ decidable and
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- All multivariate polynomials
- Many elementary transcendental fn, e.g. $\exp, \ln, \log_q, \sin, \cos, \tan, \arctan$
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Theorem

For functions in $\mathcal{F}_{\mathrm{DA}}$, checking non-linear conflicts is decidable and linearisations are computable.

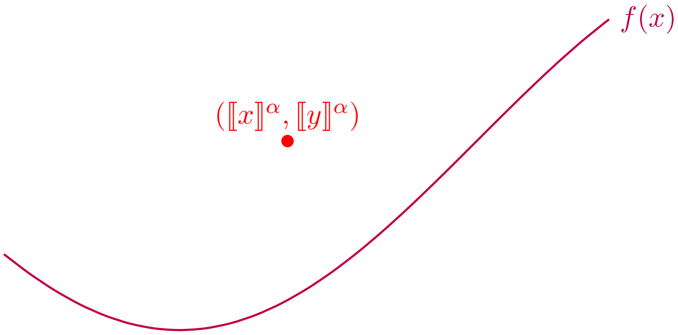
Using functions' known properties

Specialised linearisation algorithms for specific combinations of subclasses of functions $g \in \mathcal{F}_{\mathsf{DA}}$ and point of conflict:

- Differentiable g: Use Tangent Space Linearisation.
- Convex/Concave g: Derive polytope R from computability of unique intersections
- Piecewise g: Meta-class: $\operatorname{dom} g$ partitioned by linear or non-linear predicates, each with a linearisation algorithm attached.
- Rational $g(\boldsymbol{x})$: Evaluate exactly in order to determine which linearisation to use.
- Irrational $g(\boldsymbol{x})$: Bound difference from below by a rational via successive approximations by the Computable Analysis implementation `iRRAM`.

Tangent space linearisation (schematic)

$\underbrace{f(x) \geq y}_{P}$, $\alpha : V \to \mathbb{Q}$



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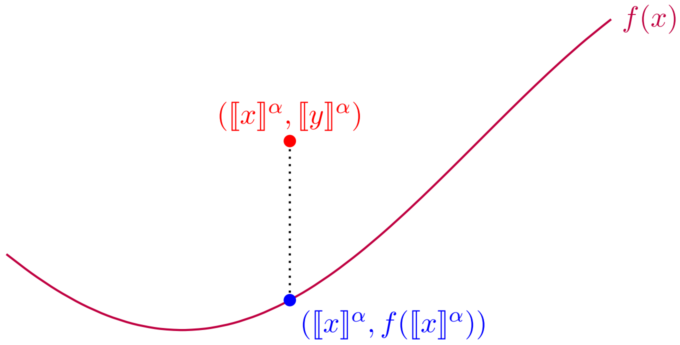

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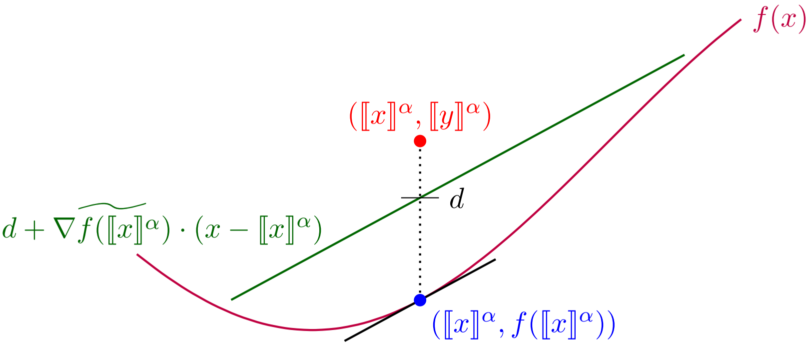

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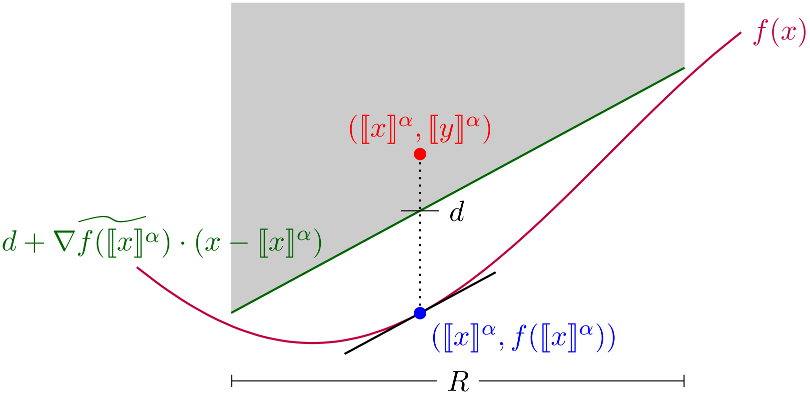
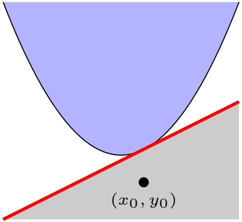
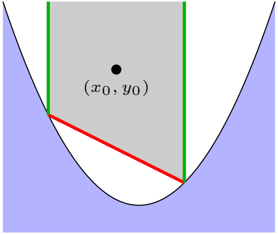


Special classes: convex/concave

- f convex:





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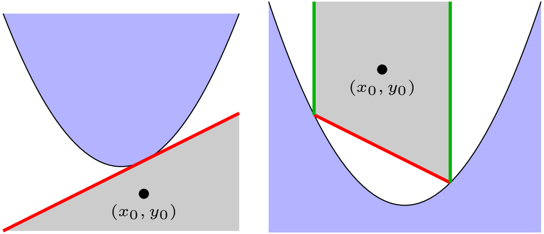

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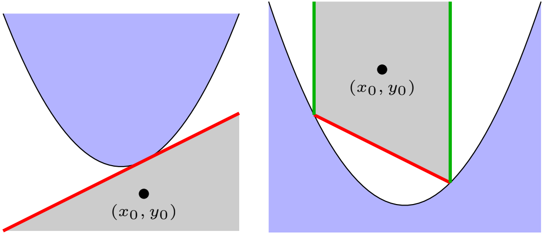

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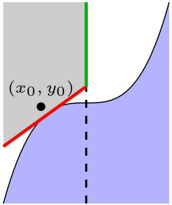
- f piecewise convex/-cave:



e.g. $x \mapsto x^{2n+1}$ for $n \in \mathbb{N}$

δ–`ksmt` termination

Sufficient termination conditions

Let $\epsilon > 0$.

- A linearisation C at an assignment α is ϵ-full if it excludes all assignments in an open ϵ-ball around α.

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A formula F is a bounded instance if

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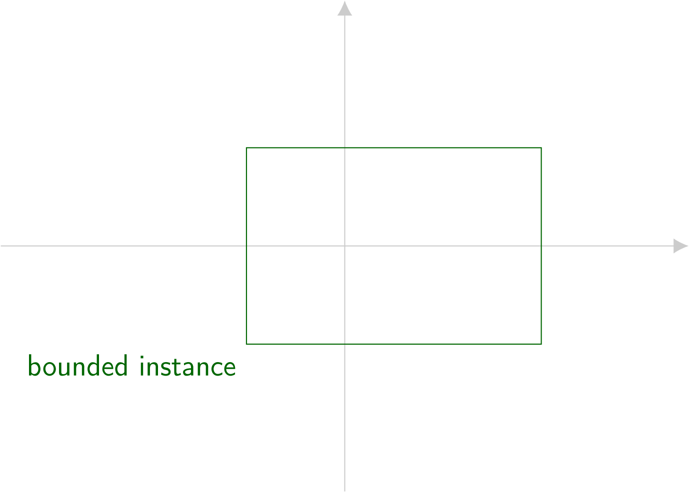
Theorem

On bounded instances, ϵ-full ksmt runs are terminating.

Terminating runs by Example



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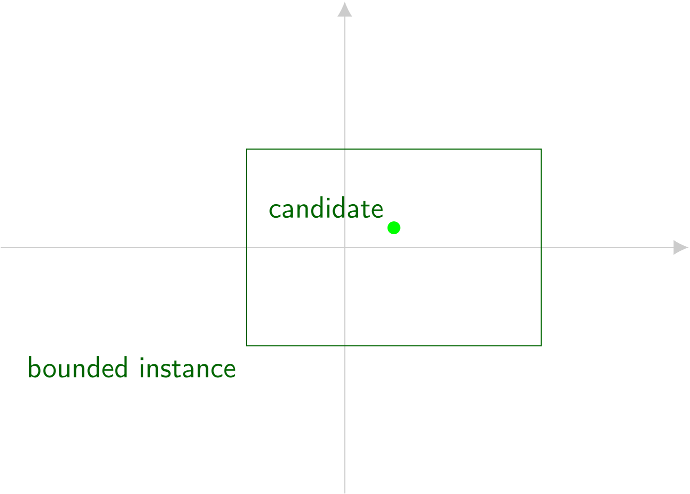

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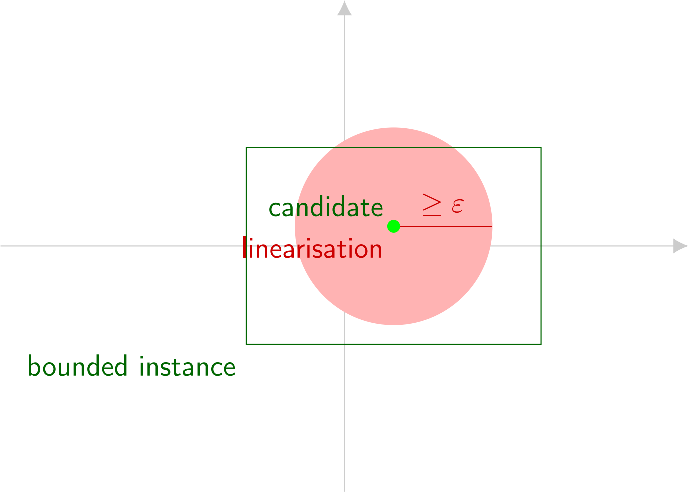

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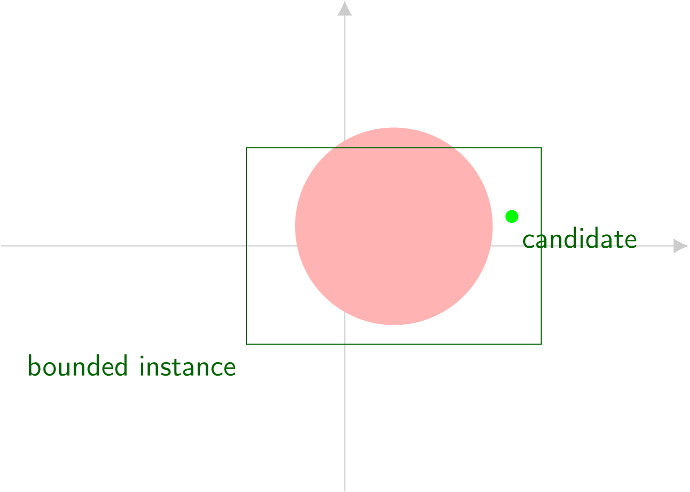

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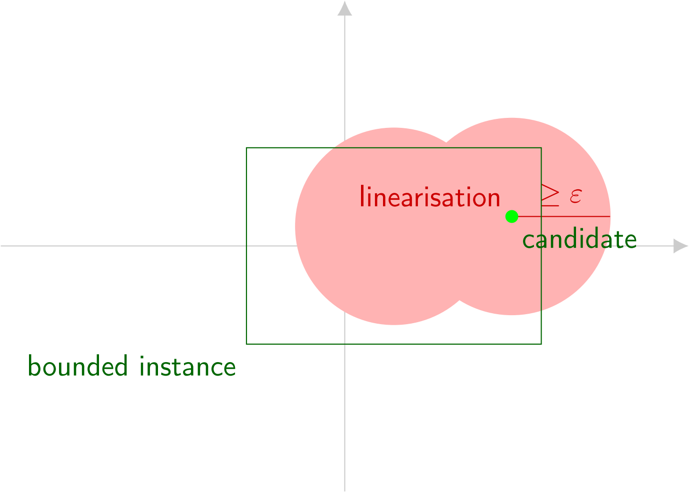

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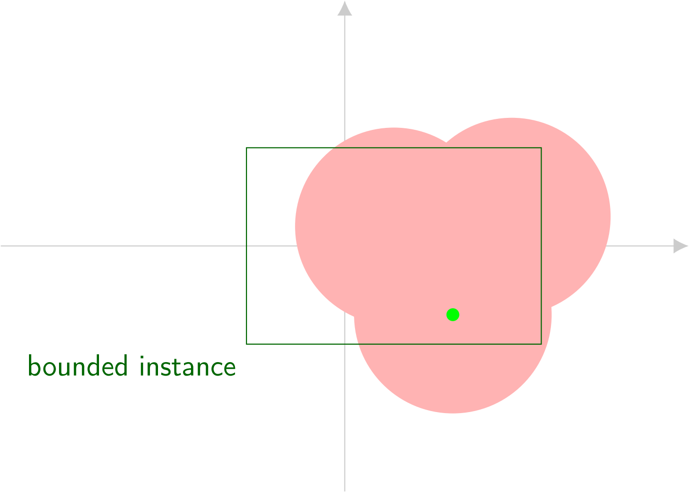

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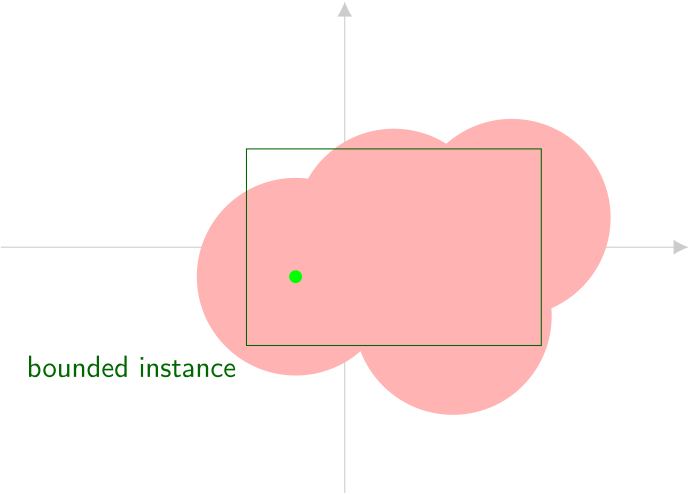

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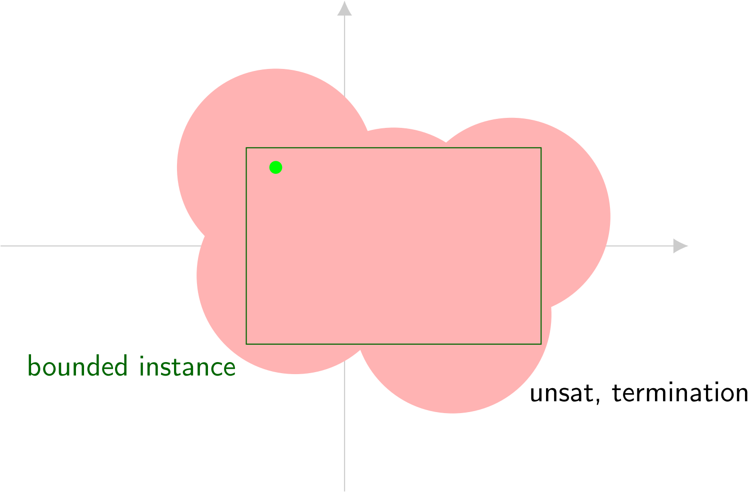
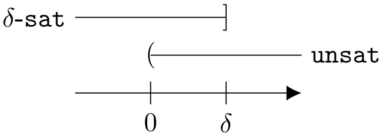


`ksmt` is a δ-complete decision procedure for non-linear constraints

δ-decidability

Let $\delta > 0$ be rational. The δ-relaxation P_δ of a constraint $P : f(\boldsymbol{x}) \diamond 0$ is

- $P_\delta : |f(\boldsymbol{x})| \leq \delta$ when $\diamond \in \{=\}$,
- $P_\delta : f(\boldsymbol{x}) \diamond \delta$ when $\diamond \in \{<, \leq\}$, and
- $P_\delta : f(\boldsymbol{x}) \diamond -\delta$ when $\diamond \in \{>, \geq\}$.



$P : f(\boldsymbol{x}) \leq 0.$

Definition [S. Gao, J. Avigad, E. Clarke, '12]

δ-deciding a formula F denotes computing

- δ-`sat`, if there is α s.t. $[\![F_\delta]\!]^\alpha =$ true.
- `unsat`, if F is unsatisfiable.

In case both answers are valid, either output is acceptable.

For `ksmt`, just relaxing the non-linear part for δ-`sat` suffices: $\mathcal{L}_0 \wedge \mathcal{N}_\delta$.

δ-ksmt calculus

- Transition rules define relation $\Rightarrow$ on states $(\alpha, \mathcal{L}, \mathcal{N})$.
  - (partial) assignment α
  - linear inequalities $\mathcal{L}$
  - non-linear units $\mathcal{N}$
- initial state is $(\text{nil}, \mathcal{L}_0, \mathcal{N}_0)$ for formula in separated linear form $\mathcal{L}_0 \wedge \mathcal{N}_0$.
- sat, $unsat$ and δ-sat are final states.

δ-ksmt calculus

Rules	$(\alpha, \mathcal{L}, \mathcal{N}) \Rightarrow \circ$	

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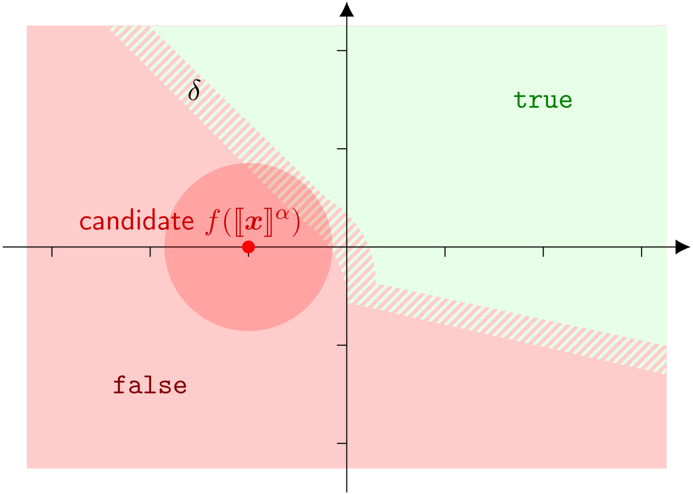
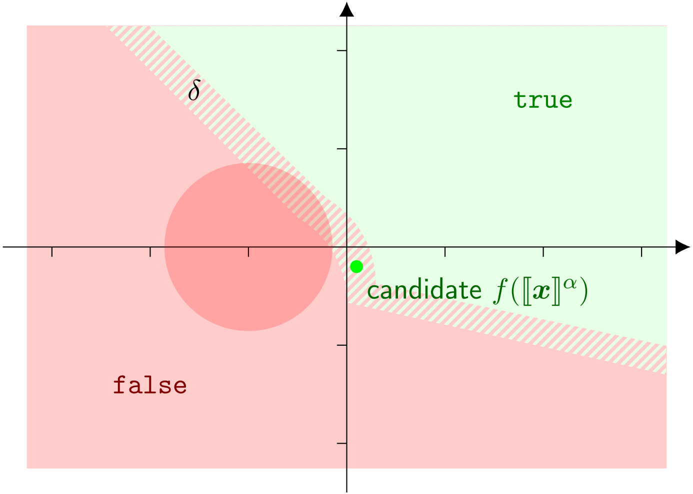
(B) backjump	$(\gamma, \mathcal{L}, \mathcal{N})$	to γ maximal conflict-free prefix of α
(L) linearise	$(\alpha, \mathcal{L} \cup L, \mathcal{N})$	L linearisation, excluding the non-linear conflict
(F^{sat})	sat	all variables are assigned, no linear conflict and (A), (R), (B), (L) are not applicable
(F^{unsat})	$unsat$	$[\![\mathcal{L}]\!]^{nil} =$ false
(F^{sat}_{δ})	δ-sat	all variables are assigned and $[\![\mathcal{L} \wedge \mathcal{N}_{\delta}]\!]^{\alpha} =$ true



δ
true
candidate $f([\![x]\!]^\alpha)$
false



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candidate $f(\llbracket \boldsymbol{x} \rrbracket^{\alpha})$
false

Theorem

Soundness of `ksmt` carries over to δ-`ksmt`.

Computable functions on $\mathbb{R}$ instead of $\mathcal{F}_{\mathsf{DA}}$ provide a computable modulus of continuity.

We provide algorithms computing ϵ-full linearisations via:

- Linearise_δ (uniform) modulus of continuity, and
- $\text{LineariseLocal}_\delta$ local modulus of continuity extracted from computability of f.

Theorem

On bounded instances, there is $\epsilon > 0$ such that δ-`ksmt` runs with linearisations computed by Linearise_δ and $\text{LineariseLocal}_\delta$ are ϵ-full.

Theorem

δ-`ksmt` is a δ-complete decision procedure.

ksmt implementation/evaluation

ksmt implementation

ksmt system:

- SMT solver for non-linear arithmetic
- Model guided architecture in the spirit of conflict resolution/MCSAT
- Including SAT/linear/non-linear in one incremental framework
- Integrates iRRAM – system for exact real arithmetic based on computable analysis developed by Norbert Th. Müller and colleagues.
- Open source: `http://informatik.uni-trier.de/~brausse/ksmt`

Conclusions and future work

`ksmt` calculus:

- model-guided search & resolution of non-linear conflicts via local linearisation
- prototypical implementation with promising results
- identified broad class of functions $\mathcal{F}_{\mathrm{DA}}$ for which conflicts are decidable
- δ-complete decision procedure for bounded instances

Future:

- more precise linearisations for specific functions
- analyze complexity of deciding conflicts