Decomposition of König's lemma and its unique variants in constructive reverse mathematics

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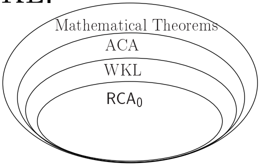
KL and WKL in Classical Reverse Mathematics

- König's lemma states that any infinite finitely-branching tree has an infinite path.
- König's lemma (over $\mathbb{N}$) KL is formalized naturally in second-order arithmetic. Weak König's lemma WKL is KL restricted to $\{0,1\}^*$-trees.
- In classical reverse mathematics (Friedman, Simpson and many others, 1970's-), many theorems in ordinary mathematics are provable in RCA_0, equivalent to WKL, or equivalent to the arithmetical comprehension ACA, which is equivalent (over RCA_0) to KL and strictly stronger than WKL.



KL and WKL in Constructive Mathematics

- In constructive/intuitionistic Mathematics (1900's-), the decidable fan theorem plays an important role (e.g. it is used to prove the uniform continuity theorem).
- The decidable fan theorem $\mathrm{FAN_D}$ is the (sort of) contrapositive of KL and its instance $\mathrm{FAN_D}(\mathrm{T}_{01})$ for $\{0,1\}^*$ is the (sort of) contrapositive of WKL.
- Nevertheless, it is widely known that $\mathrm{FAN_D}$ is constructively equivalent to $\mathrm{FAN_D}(\mathrm{T}_{01})$.
- This is because the **countable choice principle**

$$\mathrm{AC}^{0,0}: \forall x^{\mathbb{N}} \exists y^{\mathbb{N}} A(x,y) \to \exists f^{\mathbb{N}\to\mathbb{N}} \forall x^{\mathbb{N}} A(x, f(x))$$

  is accepted in constructive/intuitionistic mathematics.
- For the same reason, KL is equivalent to bounded König's lemma BKL (in Classical RM), which is derived from WKL constructively.

For the connection between classical reverse mathematics and constructive mathematics, it is valuable to see what kind of choice principle is needed to derive KL from WKL constructively.

Framework

- Our base theory EL_0 is a subsystem of intuitionistic analysis EL (Kreisel/Troelstra 1970), which has two-sorted (natural numbers/functions over natural numbers) variables in its language.
  The subscript 0 of EL_0 denotes the restriction of induction scheme to Σ^0_1-formulas $\exists x^{\mathbb{N}} A_{\mathrm{qf}}$ in this context.
- This system EL_0 is employed as a base theory in (recent) **constructive reverse mathematics**.
- One may obtain (the equivalents of) RCA_0 and RCA in **classical reverse mathematics** from EL_0 and EL respectively by adding the law of excluded middle scheme $A \vee \neg A$ into the axioms.
- EL_0 contains only

$$\mathrm{QF\text{-}AC}^{0,0} : \forall x^{\mathbb{N}} \exists y^{\mathbb{N}} A_{\mathrm{qf}}(x, y) \to \exists f^{\mathbb{N}\to\mathbb{N}} \forall x^{\mathbb{N}} A_{\mathrm{qf}}(x, f(x)).$$

Definition

- KL: For all infinite finitely-branching trees $T \subseteq \mathbb{N}^*$, there exists $p^{\mathbb{N}\to\mathbb{N}}$ s.t. $\forall i^{\mathbb{N}} (\overline{p}i \in T)$.
- BKL: For all infinite bounded trees $T \subseteq \mathbb{N}^*$, there exists $p^{\mathbb{N}\to\mathbb{N}}$ s.t. $\forall i^{\mathbb{N}} (\overline{p}i \in T)$.
- WKL: For all infinite trees $T \subseteq \{0,1\}^*$, there exists $p^{\mathbb{N}\to\mathbb{N}}$ s.t. $\forall i^{\mathbb{N}} (\overline{p}i \in T)$.

- $T \subseteq \mathbb{N}^*(\approx \mathbb{N})$ is an infinite tree if $\forall u, v\, (u \in T \wedge v \preceq u \to v \in T)$ and $\forall i^{\mathbb{N}} \exists u^{\mathbb{N}^*} (|u| = i \wedge u \in T)$.
- A tree T is **finitely-branching** if

$$\forall u \in T \exists k^{\mathbb{N}} \forall x^{\mathbb{N}} \left(u * \langle x \rangle \in T \to x \leq k\right).$$

- A tree T is **bounded** if T has a height-wise bounding function $h^{\mathbb{N}\to\mathbb{N}}$ which satisfies

$$\forall u \in T \forall j < |u|\, (u_j \leq h(j)).$$

Fact

1. $\mathsf{EL}_0 \vdash \mathrm{WKL} \leftrightarrow \mathrm{BKL}$.
2. $\mathsf{EL}_0 \vdash \mathrm{WKL} + \mathrm{BT}_{\mathrm{fb}} \rightarrow \mathrm{KL}$.

$\mathrm{BT}_{\mathrm{fb}}$: Every infinite finitely-branching tree has a height-wise bounding function.

Fact

1. $\mathsf{EL}_0 \vdash \mathrm{WKL} \leftrightarrow \mathrm{BKL}$.
2. $\mathsf{EL}_0 \vdash \mathrm{WKL} + \mathrm{BT}_{\mathrm{fb}} \to \mathrm{KL}$.

$\mathrm{BT}_{\mathrm{fb}}$: Every infinite finitely-branching tree has a height-wise bounding function.

Proposition (cf. Kohlenbach 2008, F. 2020)

$\mathsf{EL}_0 + \Pi^0_1\text{-}\mathrm{AC}^{0,0} \vdash \mathrm{BT}_{\mathrm{fb}}$.

Proposition (cf. Troelstra/van Dalen 1988, F. 2020)

$\mathsf{EL}_0 + \mathrm{B}\Pi^0_1 + \Sigma^0_2\text{-}\mathrm{IND} + \Pi^0_1\text{-}\mathrm{AC}^{0,0}! \vdash \mathrm{BT}_{\mathrm{fb}}$.

Remark. $\mathsf{RCA}_0 + \mathrm{BT}_{\mathrm{fb}} \vdash \Sigma^0_2\text{-}\mathrm{IND}$ (due to Keita Yokoyama). Note that
$\mathsf{RCA}_0 (= \mathsf{EL}_0 + \mathrm{LEM}) \subsetneq \mathsf{RCA}_0 + \mathrm{B}\Pi^0_1 \subsetneq \mathsf{RCA}_0 + \Sigma^0_2\text{-}\mathrm{IND}$.

- $\mathrm{AC}^{0,0}!$ (Countable Unique Choice Principle):

$$\forall x^{\mathbb{N}} \exists y^{\mathbb{N}} (A(x,y) \wedge \forall y' (A(x,y') \to y' = y)) \to \exists f \forall x\, A(x, f(x))$$

- $\Pi^0_1\text{-}\mathrm{AC}^{0,0}!$:

$$\forall x^{\mathbb{N}} \exists y^{\mathbb{N}} \left(\forall z^{\mathbb{N}} A_{\mathrm{qf}}(x,y,z) \wedge \forall y' (\forall z\, A_{\mathrm{qf}}(x,y',z) \to y' = y)\right)$$
$$\to \exists f \forall x, z\, A_{\mathrm{qf}}(x, f(x), z)$$

- $\mathrm{B}\Pi^0_1$:

$$\forall n\, (\forall x < n \exists y \forall z A_{\mathrm{qf}}(x,y,z) \to \exists y' \forall x < n \exists y < y' \forall z A_{\mathrm{qf}}(x,y,z)) \,.$$

Another Approach for Deriving $\mathrm{BT}_{\mathrm{fb}}$

Proposition (F. 2020)

$\mathsf{EL}_0 + \mathrm{B}\Pi^0_1 + \Sigma^0_2\text{-IND} + \mathrm{KL!} \vdash \mathrm{BT}_{\mathrm{fb}}$.

$\mathrm{KL!}$: For all infinite finitely-branching trees $T \subseteq \mathbb{N}^*$ satisfying

$$(!) : \forall p^{\mathbb{N}\to\mathbb{N}}, q^{\mathbb{N}\to\mathbb{N}} (\exists n(p(n) \neq q(n)) \to \exists n (\overline{p}n \notin T) \vee \exists n (\overline{q}n \notin T)),$$

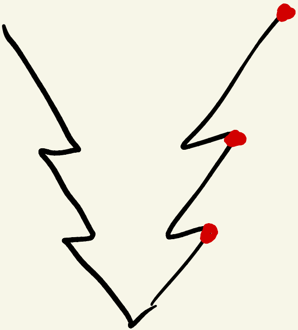
there exists $p^{\mathbb{N}\to\mathbb{N}}$ s.t. $\forall i^{\mathbb{N}} (\overline{p}i \in T)$.

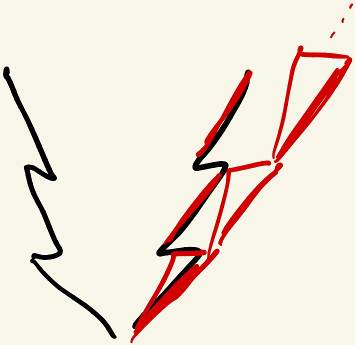
Corollary

$\mathsf{EL}_0 + \mathrm{B}\Pi^0_1 + \Sigma^0_2\text{-IND} \vdash \mathrm{WKL} + \mathrm{BT}_{\mathrm{fb}} \leftrightarrow \mathrm{KL}$.

$\Pi^0_1\text{-}AC^{0,0}! \to BT_{fb}$

$KL! \to BT_{fb}$





(Superfluous part is also contained.)

Theorem (Berger/Ishihara 2005, Schwichtenberg 2005)

$\mathrm{WKL!} \leftrightarrow \mathrm{FAN_D}(\mathrm{T_{01}})$ (which is formalizable in EL_0).

$\mathrm{WKL!}$: For any infinite tree $T \subseteq \{0,1\}^*$ s.t.

$$\forall p, q \in \{0,1\}^{\mathbb{N}} \left(\exists n(p(n) \neq q(n)) \to \exists n \left(\overline{p}n \notin T\right) \vee \exists n \left(\overline{q}n \notin T\right)\right),$$

there exists $p^{\mathbb{N}\to\mathbb{N}}$ s.t. $\forall i^{\mathbb{N}} \left(\overline{p}i \in T\right)$.

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there exists $p^{\mathbb{N}\to\mathbb{N}}$ s.t. $\forall i^{\mathbb{N}}\,(\overline{p}i \in T)$.

Remark. EL_0 proves that for any bounded tree $T^h \subseteq \mathbb{N}^*$, (!) is equivalent to the following:

$$\forall p, q \in \{f^{\mathbb{N}\to\mathbb{N}} \mid \forall i^{\mathbb{N}}\,(f(i) \leq h(i))\}$$
$$\left(\exists n(p(n) \neq q(n)) \to \exists n\,\left(\overline{p}n \notin T^h\right) \vee \exists n\,\left(\overline{q}n \notin T^h\right)\right).$$

Thus the uniqueness condition (!) is adapted in the sense of $\mathrm{WKL!}$ to any bounded tree.

Another Uniqueness Condition

- $\mathrm{WKL!!}$ (by Moschovakis 2012):

  For any infinite tree $T \subseteq \{0,1\}^*$ satisfying

$$(!!) : \forall p^{\mathbb{N}\to\mathbb{N}}, q^{\mathbb{N}\to\mathbb{N}} (\forall n (\overline{p}n \in T) \land \forall n (\overline{q}n \in T) \to \forall n(p(n) = q(n))),$$

  there exists $p^{\mathbb{N}\to\mathbb{N}}$ s.t. $\forall i^{\mathbb{N}} (\overline{p}i \in T)$.
- Define $\mathrm{BKL!!}$ and $\mathrm{KL!!}$ by using (!!) in the same manner.

Fact

Since (!) implies (!!) constructively, we have
$\mathrm{KL} \to \mathrm{KL!!} \to \mathrm{KL!}$ immediately.

Proposition (F. 2020)

$\mathsf{EL}_0 + \mathrm{MP} \vdash (!!) \to (!)$, where $\mathrm{MP} : \neg\neg\exists x A_{\mathrm{qf}} \to \exists x A_{\mathrm{qf}}$.

- The standard proof of $\mathsf{EL}_0 + \mathrm{BT}_{\mathrm{fb}} \vdash \mathrm{WKL} \to \mathrm{BKL}$ also shows $\mathsf{EL}_0 + \mathrm{BT}_{\mathrm{fb}} \vdash \mathrm{WKL!!} \to \mathrm{BKL!!}$, and hence,

  $\mathsf{EL}{+}\mathrm{MP} \vdash \mathrm{WKL!}{+}\mathrm{BT}_{\mathrm{fb}} \leftrightarrow \mathrm{WKL!!}{+}\mathrm{BT}_{\mathrm{fb}} \leftrightarrow \mathrm{KL!!} \leftrightarrow \mathrm{KL!}$.

- On the other hand, the condition (!) seems not to be preserved by the standard embedding of a bounded tree into a $\{0,1\}$-tree. Then it is not trivial whether $\mathrm{KL!}$ is equivalent to $\mathrm{WKL!} + \mathrm{BT}_{\mathrm{fb}}$ in the absence of MP.

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Lemma (Generalization of Schwichtenberg's Argument)

$\mathsf{EL}_0 \vdash \mathrm{BFAN_D} \to \mathrm{BKL!}$, *where* $\mathrm{BFAN_D}$ *is the decidable fan theorem restricted to* **bounded** *trees.*

Corollary

$\mathsf{EL}_0 + \mathrm{B\Pi^0_1} + \Sigma^0_2\text{-}\mathrm{IND} \vdash \mathrm{WKL!} + \mathrm{BT_{fb}} \leftrightarrow \mathrm{KL!}.$

Proof. $\mathrm{WKL!} + \mathrm{BT_{fb}} \to \mathrm{FAN_D}(\mathrm{T_{01}}) + \mathrm{BT_{fb}} \to \mathrm{BFAN_D} + \mathrm{BT_{fb}} \to \mathrm{BKL!} + \mathrm{BT_{fb}} \to \mathrm{KL!}.$

Proposition (cf. Troelstra/van Dalen 1988, F. 2020, Revisited)

$\mathsf{EL}_0 + \mathrm{B}\Pi^0_1 + \Sigma^0_2\text{-IND} + \Pi^0_1\text{-AC}^{0,0}! \vdash \mathrm{BT}_{\mathrm{fb}}$, and hence,

$\mathsf{EL}_0 + \mathrm{B}\Pi^0_1 + \Sigma^0_2\text{-IND} + \Pi^0_1\text{-AC}^{0,0}! \vdash \mathrm{WKL}\,(\leftrightarrow \mathrm{BKL}) \rightarrow \mathrm{KL}$.

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$\mathsf{EL}_0 + \mathrm{B}\Pi^0_1 + \Sigma^0_2\text{-IND} + \Pi^0_1\text{-AC}^{0,0}! \vdash \mathrm{WKL}\,(\leftrightarrow \mathrm{BKL}) \rightarrow \mathrm{KL}.$

Question.

Can we characterize the difference of KL and WKL by some fragment of the unique choice principle?

Decomposition of KL

For this purpose, we consider the following variant:

$$\Pi^0_1\text{-}\mathrm{AC}^{0,0}_{\mathrm{d}} : \quad \begin{array}{l} \forall x^{\mathbb{N}} \exists y^{\mathbb{N}} \left(\forall z^{\mathbb{N}} A_{\mathrm{qf}}(x,y,z) \land \forall y' (\forall z \leq x A_{\mathrm{qf}}(x,y',z) \to y' \leq y) \right) \\ \to \exists f^{\mathbb{N}\to\mathbb{N}} \forall x, z\, A_{\mathrm{qf}}(x, f(x), z). \end{array}$$

Theorem

$\mathsf{EL}_0 + \mathrm{B}\Pi^0_1 + \Sigma^0_2\text{-}\mathrm{IND} \vdash \mathrm{WKL} + \Pi^0_1\text{-}\mathrm{AC}^{0,0}_{\mathrm{d}} \leftrightarrow \mathrm{KL}$.

Remark. $\Pi^0_1\text{-}\mathrm{AC}^{0,0}_{\mathrm{d}}$ is a weakening of

$$\Pi^0_1\text{-}\mathrm{AC}^{0,0}_{\mathrm{wu}} : \quad \begin{array}{l} \forall x^{\mathbb{N}} \exists y^{\mathbb{N}} \left(\forall z^{\mathbb{N}} A_{\mathrm{qf}}(x,y,z) \land \forall y' (\forall z\, A_{\mathrm{qf}}(x,y',z) \to y' \leq y) \right) \\ \to \exists f^{\mathbb{N}\to\mathbb{N}} \forall x, z\, A_{\mathrm{qf}}(x, f(x), z), \end{array}$$

where "wu" stands for the **weakened uniqueness** condition: $\forall y' (\forall z\, A_{\mathrm{qf}}(x,y',z) \to y' \leq y)$. Note that $\Pi^0_1\text{-}\mathrm{AC}^{0,0}_{\mathrm{wu}}$ (and hence, $\Pi^0_1\text{-}\mathrm{AC}^{0,0}_{\mathrm{d}}$) is derived from $\mathrm{AC}^{0,0}$!.

Decomposition of $\mathrm{KL!!}$

Lemma

$\mathsf{EL}_0 + \mathrm{B}\Pi^0_1 + \Sigma^0_2\text{-}\mathrm{IND} + \Pi^0_1\text{-}\mathrm{AC}^{0,0}_{\mathrm{d}}! \vdash \mathrm{BT}_{\mathrm{fb}}$.

$\Pi^0_1\text{-}\mathrm{AC}^{0,0}_{\mathrm{d}}!$:

$$\forall x^{\mathbb{N}} \exists y^{\mathbb{N}} \left(\forall z^{\mathbb{N}} A_{\mathrm{qf}}(x,y,z) \wedge \forall y' \left(\begin{array}{l} (\forall z A_{\mathrm{qf}}(x,y',z) \to y' = y) \wedge \\ (\forall z \leq x A_{\mathrm{qf}}(x,y',z) \to y' \leq y) \end{array} \right)\right)$$
$$\to \exists f^{\mathbb{N}\to\mathbb{N}} \forall x, z\, A_{\mathrm{qf}}(x, f(x), z);$$

Remark. $\Pi^0_1\text{-}\mathrm{AC}^{0,0}_{\mathrm{d}}!$ is an obvious weakening of $\Pi^0_1\text{-}\mathrm{AC}^{0,0}_{\mathrm{d}}$ and also of $\Pi^0_1\text{-}\mathrm{AC}^{0,0}!$.

Theorem

$\mathsf{EL}_0 + \mathrm{B}\Pi^0_1 + \Sigma^0_2\text{-}\mathrm{IND} \vdash \mathrm{WKL!!} + \Pi^0_1\text{-}\mathrm{AC}^{0,0}_{\mathrm{d}}! \leftrightarrow \mathrm{KL!!}$.

$$
\begin{tikzcd}
 & \Pi^0_1\text{-}\mathrm{AC}^{0,0}_{\mathrm{wu}} \arrow[dl] \arrow[d] \\
\Pi^0_1\text{-}\mathrm{AC}^{0,0}_{\mathrm{d}} \arrow[d] & \Pi^0_1\text{-}\mathrm{AC}^{0,0}! \arrow[dl] \\
\Pi^0_1\text{-}\mathrm{AC}^{0,0}_{\mathrm{d}}! \arrow[d] & \\
\mathrm{BT}_{\mathrm{fb}} &
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & \Pi^0_1\text{-}\mathrm{AC}^{0,0}_{\mathrm{wu}} \arrow[dl] \arrow[d] \\
\Pi^0_1\text{-}\mathrm{AC}^{0,0}_{\mathrm{d}} \arrow[d] & \Pi^0_1\text{-}\mathrm{AC}^{0,0}! \arrow[dl] \\
\Pi^0_1\text{-}\mathrm{AC}^{0,0}_{\mathrm{d}}! \arrow[d] & \\
\mathrm{BT}_{\mathrm{fb}} &
\end{tikzcd}
$$

Question

Can we characterize $\mathrm{BT}_{\mathrm{fb}}$ by a unique choice principle without using notions on trees?

Decomposition of $\mathrm{KL}!$

For this purpose, we consider (a weakening of) the **bounding choice principle**

$$\mathrm{BC}^{0,0} : \forall x^{\mathbb{N}} \exists y^{\mathbb{N}} A(x,y) \to \exists g^{\mathbb{N}\to\mathbb{N}} \forall x^{\mathbb{N}} \exists y \leq g(x) A(x,y).$$

In particular, we introduce the Π^0_1-fragment of $\mathrm{BC}^{0,0}$ with a **strengthened uniqueness** condition as follows:

$$\Pi^0_1\text{-}\mathrm{BC}^{0,0}_{\mathrm{su}} : \begin{array}{l} \forall x^{\mathbb{N}} \exists y^{\mathbb{N}} \left(\forall z^{\mathbb{N}} A_{\mathrm{qf}}(x,y,z) \wedge \forall y' \left(\forall z \leq y A_{\mathrm{qf}}(x,y',z) \to y' = y \right) \right) \\ \to \exists g^{\mathbb{N}\to\mathbb{N}} \forall x^{\mathbb{N}} \exists y \leq g(x) \forall z^{\mathbb{N}} A_{\mathrm{qf}}(x,y,z). \end{array}$$

Remark. $\Pi^0_1\text{-}\mathrm{BC}^{0,0}_{\mathrm{su}}$ is derived from

$$\Pi^0_1\text{-}\mathrm{AC}^{0,0}_{\mathrm{su}} : \begin{array}{l} \forall x^{\mathbb{N}} \exists y^{\mathbb{N}} \left(\forall z^{\mathbb{N}} A_{\mathrm{qf}}(x,y,z) \wedge \forall y' \left(\forall z \leq y A_{\mathrm{qf}}(x,y',z) \to y' = y \right) \right) \\ \to \exists f^{\mathbb{N}\to\mathbb{N}} \forall x, z\, A_{\mathrm{qf}}(x,f(x),z), \end{array}$$

which is derived from $\Pi^0_1\text{-}\mathrm{AC}^{0,0}!$.

Theorem

$\mathsf{EL}_0 + \mathrm{B}\Pi^0_1 + \Sigma^0_2\text{-IND} \vdash \mathrm{BT}_{\mathrm{fb}} \leftrightarrow \Pi^0_1\text{-BC}^{0,0}_{\mathrm{su}}$, and hence,
$\mathsf{EL}_0 + \mathrm{B}\Pi^0_1 + \Sigma^0_2\text{-IND} \vdash \mathrm{WKL!} + \Pi^0_1\text{-BC}^{0,0}_{\mathrm{su}} \leftrightarrow \mathrm{KL!}$.

Theorem

$\mathsf{EL}_0 + \mathrm{B}\Pi^0_1 + \Sigma^0_2\text{-}\mathrm{IND} \vdash \mathrm{BT}_{\mathrm{fb}} \leftrightarrow \Pi^0_1\text{-}\mathrm{BC}^{0,0}_{\mathrm{su}}$, and hence,
$\mathsf{EL}_0 + \mathrm{B}\Pi^0_1 + \Sigma^0_2\text{-}\mathrm{IND} \vdash \mathrm{WKL!} + \Pi^0_1\text{-}\mathrm{BC}^{0,0}_{\mathrm{su}} \leftrightarrow \mathrm{KL!}$.

$$
\begin{tikzcd}
\mathrm{AC}^{0,0} \arrow[r] \arrow[d] & \mathrm{AC}^{0,0}! \arrow[d] \\
\Pi^0_1\text{-}\mathrm{AC}^{0,0} \arrow[r] & \Pi^0_1\text{-}\mathrm{AC}^{0,0}_{\mathrm{wu}} \arrow[dl] \arrow[d] \\
\Pi^0_1\text{-}\mathrm{AC}^{0,0}_{\mathrm{d}} \arrow[d] & \Pi^0_1\text{-}\mathrm{AC}^{0,0}! \arrow[dl] \arrow[d] \\
\Pi^0_1\text{-}\mathrm{AC}^{0,0}_{\mathrm{d}}! \arrow[d] & \Pi^0_1\text{-}\mathrm{AC}^{0,0}_{\mathrm{su}} \arrow[d] \\
\mathrm{BT}_{\mathrm{fb}} \arrow[r, leftrightarrow] & \Pi^0_1\text{-}\mathrm{BC}^{0,0}_{\mathrm{su}}
\end{tikzcd}
$$

Theorem

$\mathsf{EL}_0 + \mathrm{B}\Pi^0_1 + \Sigma^0_2\text{-IND} \vdash \mathrm{BT}_{\mathrm{fb}} \leftrightarrow \Pi^0_1\text{-BC}^{0,0}_{\mathrm{su}}$, and hence,
$\mathsf{EL}_0 + \mathrm{B}\Pi^0_1 + \Sigma^0_2\text{-IND} \vdash \mathrm{WKL!} + \Pi^0_1\text{-BC}^{0,0}_{\mathrm{su}} \leftrightarrow \mathrm{KL!}$.

$$
\begin{tikzcd}
 & \mathrm{AC}^{0,0} \arrow[r] \arrow[d] & \mathrm{AC}^{0,0}! \arrow[d] \\
 & \Pi^0_1\text{-AC}^{0,0} \arrow[r] & \Pi^0_1\text{-AC}^{0,0}_{\mathrm{wu}} \arrow[dl] \arrow[d] \\
\mathrm{KL} \arrow[r] \arrow[d] & \Pi^0_1\text{-AC}^{0,0}_{\mathrm{d}} \arrow[d] & \Pi^0_1\text{-AC}^{0,0}! \arrow[dl] \arrow[d] \\
\mathrm{KL!!} \arrow[r] \arrow[d] & \Pi^0_1\text{-AC}^{0,0}_{\mathrm{d}}! \arrow[d] & \Pi^0_1\text{-AC}^{0,0}_{\mathrm{su}} \arrow[d] \\
\mathrm{KL!} \arrow[r] & \mathrm{BT}_{\mathrm{fb}} \arrow[r, leftrightarrow] & \Pi^0_1\text{-BC}^{0,0}_{\mathrm{su}}
\end{tikzcd}
$$

Remarks

- We have decomposed KL, $\mathrm{KL!!}$ and $\mathrm{KL!}$ into their binary variants and some unique choice principles respectively.
- On the other hand, it is known that WKL can be decomposed into a logical principle (Σ^0_1-DML) and a choice principle (Π^0_1-$\mathrm{AC}^\vee$). Hence, KL is decomposed into a logical principle and two choice principles.
- We have a similar decomposition for $\mathrm{WKL!!}$ recently. By using the decomposition, it follows that $\mathrm{KL!!}$ is decomposed into two logical principles and three choice principles.
- In contrast, we only know a decomposition of $\mathrm{WKL!}$ by a logical principle and a choice principle in terms of some notions on trees. It is a remained question how $\mathrm{WKL!}$ (equivalently $\mathrm{FAN_D}(\mathrm{T_{01}})$) can be decomposed into logical and choice principles without using the notions on trees.

On Proofs

- Our proofs are modifications of the arguments in Berger/Ishihara/Schuster 2012, which was developed for the direct proof of the above mentioned decomposition of WKL.
- In particular, for deriving the unique choice principle in question from the corresponding variant of KL, we consider the induced **dependent choice principles**.
- Here we present the proof of $\mathrm{KL} \to \Pi^0_1\text{-}\mathrm{AC}^{0,0}_{\mathrm{d}}$ as a sample.

Proof of $\mathrm{KL} \to \Pi^0_1\text{-}\mathrm{AC}^{0,0}_{\mathrm{d}}$

Recall that $\Pi^0_1\text{-}\mathrm{AC}^{0,0}_{\mathrm{d}}$ is the following principle:

$$\forall x^{\mathbb{N}} \exists y^{\mathbb{N}} \left(\forall z^{\mathbb{N}} A_{\mathrm{qf}}(x, y, z) \wedge \forall y' \left(\forall z \leq x\, A_{\mathrm{qf}}(x, y', z) \to y' \leq y\right)\right)$$
$$\to \exists f^{\mathbb{N} \to \mathbb{N}} \forall x, z\, A_{\mathrm{qf}}(x, f(x), z).$$

The induced dependence choice principle $\Pi^0_1\text{-}\mathrm{DC}^{0,0}_{\mathrm{d}}$ is the following:

$$\forall u \in \mathbb{N}^* \exists y^{\mathbb{N}} \left(\forall z^{\mathbb{N}} A'_{\mathrm{qf}}(u, y, z) \wedge \forall y' \left(\forall z \leq |u| A'_{\mathrm{qf}}(u, y', z) \to y' \leq y\right)\right)$$
$$\to \exists f^{\mathbb{N} \to \mathbb{N}} \forall x, z\, A'_{\mathrm{qf}}(\overline{f} x, f(x), z);$$

By taking $A'_{\mathrm{qf}}(u, y, z) :\equiv A_{\mathrm{qf}}(|u|, y, z)$, we have that $\Pi^0_1\text{-}\mathrm{DC}^{0,0}_{\mathrm{d}}$ implies $\Pi^0_1\text{-}\mathrm{AC}^{0,0}_{\mathrm{d}}$.
Then it suffices to show that KL implies $\Pi^0_1\text{-}\mathrm{DC}^{0,0}_{\mathrm{d}}$.

Lemma

$\mathrm{EL}_0 + \Sigma^0_2\text{-IND} + \mathrm{KL} \vdash \Pi^0_1\text{-DC}^{0,0}_{\mathrm{d}}$.

Proof. (cf. Berger/Ishihara/Schuster 2012)
Let $A_{\mathrm{qf}}(u, i, k)$ satisfy

$$\forall u \in \mathbb{N}^* \exists i^{\mathbb{N}} \left(\forall k^{\mathbb{N}} A_{\mathrm{qf}}(u, i, k) \wedge \forall j^{\mathbb{N}} \left(\forall k \leq |u| A_{\mathrm{qf}}(u, j, k) \to j \leq i\right)\right). \tag{1}$$

Define $T \subseteq \mathbb{N}^*$ as $u \in T$ if and only if

$$\forall n < |u| \forall k < |u| A_{\mathrm{qf}}(\overline{u}n, u_n, k).$$

Then T is trivially a tree. For verifying that T is infinite, one can show $\forall i \exists u \in \mathbb{N}^i \forall n < |u| \forall k\, A_{\mathrm{qf}}(\overline{u}n, u_n, k)$ by Σ^0_2-induction on i.

To show that T is finitely-branching, fix $u \in T$. By (1), there exists $i^{\mathbb{N}}$ such that $\forall j^{\mathbb{N}} \left(\forall k \leq |u| A_{\mathrm{qf}}(u, j, k) \to j \leq i\right)$. If $u * \langle x \rangle \in T$, then $\forall k < |u * \langle x \rangle| A_{\mathrm{qf}}(u, x, k)$, and hence, $x \leq i$.

By KL, there exists a path $p^{\mathbb{N}\to\mathbb{N}}$ through T.

In the following, we claim $\forall n^{\mathbb{N}}, k^{\mathbb{N}} A_{\mathrm{qf}}(\overline{p}n, p(n), k)$.
Fix $n^{\mathbb{N}}$ and $k^{\mathbb{N}}$.
Let $i' := n + k + 1$. Then $n, k < i'$.
Since p is a path through T, we have $\overline{p}\, i' \in T$, and hence,
$A_{\mathrm{qf}}\left(\overline{(\overline{p}\, i')}n, (\overline{p}\, i')_n, k\right)$, namely, $A_{\mathrm{qf}}(\overline{p}n, p(n), k)$. $\square$

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