Some Lifschitz-like realizability notions separating non-constructive principles

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What is ... reverse mathematics?

- Reverse mathematics is a program to determine the exact (set-existence) axioms which are needed to prove theorems of ordinary mathematics.
- Usually, we employ a subsystem $\mathbf{RCA_0}$ of second order arithmetic as our base system, which consists of:
  1. Basic first-order arithmetic (e.g. the first-order theory of the non-negative parts of discretely ordered rings).
  2. Σ^0_1-induction schema.
  3. Δ^0_1-comprehension schema.
- Roughly speaking, $\mathbf{RCA_0}$ corresponds to (non-uniform) computable mathematics (as Δ^0_1 = computable).

Nonconstructive Principles 1

In this talk, we consider the following principles:

- The lessor limited principle of omniscience **LLPO** states that for any regular Cauchy real x, either $x \leq 0$ or $x \geq 0$.
- The binary expansion principle **BE** states that every regular Cauchy real has a binary expansion.
- The intermediate value theorem **IVT** states that for any continuous function $f : [0, 1] \to [-1, 1]$ if $f(0)$ and $f(1)$ have different signs then there is a regular Cauchy real $x \in [0, 1]$ such that $f(x) = 0$.
- Weak König's lemma **WKL** states that every infinite binary tree has an infinite path.

Here, a *regular Cauchy real* is a real x which is represented by a sequence $(q_n)_{n \in \omega}$ of rational numbers such that

$$|q_n - q_m| < 2^{-n} \text{ for any } m \geq n.$$

Constructive Reverse Mathematics 1

- LLPO, BE, IVT, etc. are considered as a non-constructive principle.
- Nevertheless, LLPO, BE, IVT are provable in RCA_0.
- In this sense, RCA_0 is too strong to be adopted as a base system.

- In order to resolve this issue, it has been proposed to replace the base system with a more constructive one.
- This proposal evolved into what is now known as Constructive Reverse Mathematics (Ishihara, and others).

- Some adopts a formalized version **BISH** of Bishop's constructive mathematics as a base system of constructive reverse mathematics.
- However, **BISH** $\vdash$ **LLPO** $\leftrightarrow$ **WKL**.
- On the other hand, $\mathbf{RCA_0} \nvdash$ **LLPO** $\leftrightarrow$ **WKL**.
- This makes it difficult to compare the results of two Reverse Math.

- $BISH$ is incomparable with RCA_0.
- Want a constructive system which is weaker than RCA_0.

Troelstra's EL_0

- $EL_0 +$ the law of excluded middle = RCA_0.
- $EL_0 +$ the axiom of countable choice = $BISH$.
- EL_0 : subsystem of RCA_0 and $BISH$.

- **RCA$_0$ $\vdash$ WKL $\not\leftrightarrow$ IVT $\leftrightarrow$ BE $\leftrightarrow$ LLPO**.
- **BISH $\vdash$ WKL $\leftrightarrow$ IVT $\leftrightarrow$ BE $\leftrightarrow$ LLPO**

Theorem (Berger-Ishihara-K.-Nemoto 2019)

EL$_0$ proves **WKL $\rightarrow$ IVT $\rightarrow$ BE $\rightarrow$ LLPO**.

[Question] Do the converse implications also hold?

Nonconstructive Principles 2

Markov's principle (double negation elimination for Σ^0_1-formulas) $\iff$

$$(\forall x, y \in [0,1])\ [y \neq 0 \ \rightarrow\ (\exists z \in \mathbb{R})\ z = x/y].$$

For a real y, "$y = 0$ or not" is non-constructive:

$$\textbf{LPO} \iff (\forall y \in \mathbb{R})\ [y = 0 \ \vee\ y \neq 0].$$

The *robust division* principle **RDIV**:

$$(\forall x, y \in [0,1])\ [x \leq y \rightarrow (\exists z \in [0,1])\ x = yz].$$

For reals x, y, "$x \leq y$ or not" is non-constructive (LLPO), but we can always replace x with $\min\{x, y\}$ without losing anything.

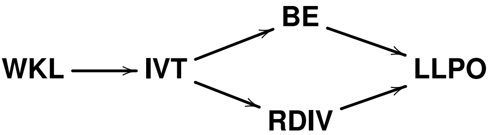
$$\textbf{RDIV} \iff (\forall x, y \in [0,1])(\exists z \in [0,1])\ \min\{x, y\} = yz.$$

Constructive Reverse Mathematics (3)

The principle **RDIV** is known to be related to

- The existence of Nash equilibria in bimatrix games.
- Executing Gaussian elimination, etc.

The following implications are known:



Question (Ishihara? Nemoto?)

Are there any other implications in the above diagram?

We will see that the above diagram is complete, via some modifications of *Lifshitz realizability*.

Realizability (1)

- A partial magma is a pair $(M,*)$ of a set M and a partial binary operation $*$ on M.
- We often write xy instead of $x*y$, and as usual, we consider $*$ as a left-associative operation, that is, xyz stands for $(xy)z$.

[Example] Define $e*n=\varphi_e(n)$. Then $(\mathbb{N},*)$ is a called *Kleene's first algebra*.

- A partial magma $(M,*)$ is *combinatory complete* if, for any term $t(x_1,x_2,\dots,x_n)$, there is $a_t\in M$ such that $a_t x_1 x_2\dots x_{n-1}\downarrow$ and $a_t x_1 x_2\dots x_n\simeq t(x_1,x_2,\dots,x_n)$.
- For terms $t(x,y)=x$, and $u(x,y,z)=xz(yz)$, the corresponding elements $a_t,a_u\in M$ are usually written as **k** and **s**.
- A combinatory complete partial magma is called a *partial combinatory algebra* (abbreviated as *pca*).

[Example] Kleene's first algebra is a pca.

Realizability (2)

- A *relative pca* is a triple $\mathbb{P} = (P, \underset{\sim}{\mathbf{P}}, *)$ such that $P \subseteq \underset{\sim}{\mathbf{P}}$, both $(\underset{\sim}{\mathbf{P}}, *)$ and $(P, * \upharpoonright P)$ are pcas, and share combinators **s** and **k**.
- In this talk, the boldface algebra $\underset{\sim}{\mathbf{P}}$ is always the set ω^ω of all infinite sequences.

In descriptive set theory, the idea of a relative pca is ubiquitous, which usually occurs as a pair of *lightface* and *boldface* pointclasses.

By the *good parametrization lemma* in descriptive set theory:

- Any $\mathbf{\Sigma}^*$-pointclass (so Spector pointclass) $\mathbf{\Gamma}$ yields a relative pca.
- The *partial* $\mathbf{\Gamma}$*-computable function application* form a lightface pca.
- The *partial* $\underset{\sim}{\mathbf{\Gamma}}$*-measurable function application* form a boldface pca.

[Example 1] If $\mathbf{\Gamma} = \mathbf{\Sigma}^0_1$:

- The induced lightface pca is equivalent to *Kleene's first algebra*.
- The boldface pca is *Kleene's second algebra*.
- The induced relative pca is known as the *Kleene-Vesley algebra*.

Realizability (2)

- A *relative pca* is a triple $\mathbb{P} = (P, \underset{\sim}{\mathbf{P}}, *)$ such that $P \subseteq \underset{\sim}{\mathbf{P}}$, both $(\underset{\sim}{\mathbf{P}}, *)$ and $(P, * \restriction P)$ are pcas, and share combinators **s** and **k**.
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[Example 2] $\mathbf{\Pi}^1_1$ is the best-known example of a Spector pointclass.

- The induced lightface pca obviously yields hyperarithmetical realizability.
- For the boldface pca, the associated total realizable functions are exactly the Borel measurable functions.

Realizability (2)

- A *relative pca* is a triple $\mathbb{P} = (P, \underset{\sim}{P}, *)$ such that $P \subseteq \underset{\sim}{P}$, both $(\underset{\sim}{P}, *)$ and $(P, * \upharpoonright P)$ are pcas, and share combinators **s** and **k**.
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In descriptive set theory, the idea of a relative pca is ubiquitous, which usually occurs as a pair of *lightface* and *boldface* pointclasses.

By the *good parametrization lemma* in descriptive set theory:

- Any Σ^*-pointclass (so Spector pointclass) Γ yields a relative pca.
- The *partial Γ-computable function application* form a lightface pca.
- The *partial $\underset{\sim}{\Gamma}$-measurable function application* form a boldface pca.

[Example 3] The infinite time Turing machines (ITTMs) form a Spector pointclass.

- ITTM-realizability has been studied by Andrej Bauer.

Lifschitz realizability

- Lifschitz (and van Oosten) used *multifunction applications*, instead of single-valued applications, to realize $\mathbf{CT_0! + \neg CT_0}$.
- Regard Π^0_1 *classes* as basic concepts rather than computable functions.

- (Lifschitz 1979) Over the Kleene first algebra $(\omega, *)$, consider the partial multifunction $\mathbf{j_L} :\subseteq \omega \rightrightarrows \omega$ defined by

$$\mathbf{j_L}(\langle e, b\rangle) = \{n \in \omega : n < b \ \land \ e * n \uparrow\},$$

  where $\mathbf{j_L}(\langle b, e\rangle)\downarrow$ if and only if the set is nonempty.
- $\mathbf{j_L}$ gives a numbering of all bounded Π^0_1 subsets of ω.
- (Van Oosten 1990) Over the Kleene second algebra $(\omega^\omega, *)$, consider the following partial multifunction $\mathbf{j_{vO}} :\subseteq \omega^\omega \rightrightarrows \omega^\omega$:

$$\mathbf{j_{vO}}(\langle g, h\rangle) = \{x \in \omega^\omega : (\forall n \in \omega)\ x(n) < h(n) \ \land \ g * x \uparrow\},$$

- $\mathbf{j_{vO}}$ gives a representation of all compact subsets of ω^ω.

Assume that $\mathbb{P} = (P, \underset{\sim}{\mathbf{P}})$ is a relative pca.

$\mathbf{j} :\subseteq \underset{\sim}{\mathbf{P}} \rightrightarrows \underset{\sim}{\mathbf{P}}$ is an *idempotent jump operator on* $\mathbb{P}$ if

1. There is $u \in P$ such that for any $a, x \in \underset{\sim}{\mathbf{P}}$, $a\mathbf{j}(x) = \mathbf{j}(uax)$.
2. There is $\eta \in P$ such that for any $x \in \underset{\sim}{\mathbf{P}}$, $x = \mathbf{j}(\eta x)$.
3. There is $\mu \in P$ such that for any $x \in \underset{\sim}{\mathbf{P}}$, $\mathbf{jj}(x) = \mathbf{j}(\mu x)$.

Here, the definition of the composition of multifunctions is:

$$hg(x) = h \circ g(x) = \begin{cases} \bigcup\{h(y) : y \in g(x)\} & \text{if } g(x) \downarrow\subseteq \mathbf{dom}(h), \\ \uparrow & \text{otherwise.} \end{cases}$$

Also, if f is a multifunction on $\underset{\sim}{\mathbf{P}}$ and $a, x \in \underset{\sim}{\mathbf{P}}$, then define $af(x) = \{ay : y \in f(x)\}$.

[Example] $\mathbf{j}_L$ and $\mathbf{j}_{\nu O}$ are idempotent jump operators.

$\mathbf{j} :\subseteq \underset{\sim}{\mathbf{P}} \rightrightarrows \underset{\sim}{\mathbf{P}}$ is an *idempotent jump operator on* $\mathbb{P}$ if

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3. There is $\mu \in P$ such that for any $x \in \underset{\sim}{\mathbf{P}}$, $\mathbf{jj}(x) = \mathbf{j}(\mu x)$.

We say that $f :\subseteq \underset{\sim}{\mathbf{P}} \rightrightarrows \underset{\sim}{\mathbf{P}}$ is *naïvely* $(P, \mathbf{j})$*-realizable* if

$$(\exists a \in P)(\forall x \in \mathbf{dom}(f))\ \mathbf{j}(ax) \subseteq f(x).$$

Remark

- This notion (for operations satisfying (1) and (2)) is implicitly studied in the work on the jump of a represented space, e.g. by de Brecht.
- One may think of $\mathbf{j}$ as an endofunctor on the category $\mathbf{Rep}$ of represented spaces and realizable functions.
- Any idempotent jump operator $\mathbf{j}$ yields a *monad* on the category $\mathbf{Rep}$: (2) monad unit (3) monad multiplication.
- Thus, the naïve $\mathbf{j}$-realizable functions on represented spaces are exactly the *Kleisli morphisms* for this monad.

- Let $\mathbf{j}$ be an idempotent jump on a relative pca $\mathbb{P} = (P, \underset{\sim}{\mathbf{P}}, *)$
- Define a new partial application $*_\mathbf{j}$ on $\underset{\sim}{\mathbf{P}}$ defined by

$$a *_\mathbf{j} b \simeq \begin{cases} a' * b & \text{if } \mathbf{j}(a) = \{a'\} \\ \uparrow & \text{otherwise} \end{cases}$$

- Hereafter, we always write a' for the unique element of $\mathbf{j}(a)$ whenever $\mathbf{j}(a)$ is a singleton. Then, consider the following:

$$P_\mathbf{j} = \{a' : a \in P \text{ and } \mathbf{j}(a) \text{ is a singleton}\}.$$

Lemma

$\mathbb{P}_\mathbf{j} = (P_\mathbf{j}, \underset{\sim}{\mathbf{P}}, *_\mathbf{j})$ is a relative pca.

In the later slides, we will define the notion of *j-realizability* and then:

Theorem

If j is an idempotent jump operator on $\mathbb{P}$,
then all axioms of **IZF** are j-realizable over $\mathbb{P}_j$.

- We say that *f is Weihrauch reducible to g* (written $f \leq_W g$) if there are partial computable functions H and K such that for any $x \in \mathrm{dom}(f)$, $y \in g(H(x))$ implies $K(x,y) \in f(x)$.
- The definition of Weihrauch reducibility $f \leq_W g$ can be viewed as the following perfect information two-player game:

$$\begin{array}{lllll} \mathrm{I}: & x_0 \in \mathrm{dom}(f) & & x_1 \in g(y_0) & \\ \mathrm{II}: & & y_0 \in \mathrm{dom}(g) & & y_1 \in f(x_0) \end{array}$$

  - Each player chooses an element from ω^ω at each round.
  - Player II wins if there is a computable strategy τ for II which yields a play described above.
- Note that y_0 depends on x_0, and y_1 depends on x_0 and x_1, and a computable strategy τ for II yields partial computable maps $H: x_0 \mapsto y_0$ and $K: (x_0, x_1) \mapsto y_1$.
- Usually, H is called an *inner reduction* and K is called an *outer reduction*.
- If reductions H and K are allowed to be continuous, then we say that *f is continuously Weihrauch reducible to g* (written $f \leq_W^c g$).

(Hirschfeldt-Jockusch 2016) For $f, g :\subseteq \omega^\omega \rightrightarrows \omega^\omega$, let us consider the following perfect information two-player game $G(f, g)$:

I:	x_0		$x_1 \in g(y_0)$		$x_2 \in g(y_1)$		$\dots$
II:		(a_0, y_0)		(a_1, y_1)		(a_2, y_2)	$\dots$

More precisely, each player chooses an element from ω^ω at each round. Here, Players I and II need to obey the following rules.

- First, Player I chooses $x_0 \in \mathbf{dom}(f)$.
- At the nth round, Player II reacts with $z_n = (a_n, y_n)$.
  - The choice $a_n = 0$ indicates that Player II makes a new query y_n to g. In this case, we require $y_n \in \mathbf{dom}(g)$.
  - The choice $a_n = 1$ indicates that Player II declares victory with y_n.
- At the $(n+1)$st round, Player I responds to the query made by Player II at the previous stage. This means that $x_{n+1} \in g(y_n)$.

Then, *Player II wins the game $G(f, g)$* if

- either Player I violates the rule before Player II violates the rule
- or Player II obeys the rule and declares victory with $y_n \in f(x_0)$.

- Player II's strategy is a code τ of a partial *continuous* function $h_\tau :\subseteq (\omega^\omega)^{<\omega} \to \omega^\omega$.
- On the other hand, Player I's strategy is any partial function $\sigma :\subseteq (\omega^\omega)^{<\omega} \to \omega^\omega$ (which is not necessarily continuous).
- Player II's strategy τ is *winning* if Player II wins along (σ, τ) whatever Player I's strategy σ is.
- We say that *f is generalized Weihrauch reducible to g* if Player II has a computable winning strategy for $G(f, g)$. In this case, we write $f \leq_{\ni \mathrm{W}} g$.
- If Player II has a (continuous) winning strategy for $G(f, g)$, we write $f \leq^c_{\ni \mathrm{W}} g$.

Lemma (Hirschfeldt-Jockusch 2016)

The relation $\leq_{\ni \mathrm{W}}$ is transitive.

- Note that the rule of the above game does not mention f except for Player I's first move. Hence, if we skip Player I's first move, we can judge if a given play follows the rule without specifying f.
- For $g :\subseteq \omega^\omega \rightrightarrows \omega^\omega$, we define $g^{\circlearrowleft} :\subseteq \omega^\omega \rightrightarrows \omega^\omega$ as follows:
  - $(x_0, \tau) \in \mathrm{dom}(g^{\circlearrowleft}) \iff \tau$ is Player II's strategy, and for Player I's any strategy σ with first move x_0, Player II declares victory at some round along (σ, τ).
  - $y \in g^{\circlearrowleft}(x_0, \tau) \iff$ Player II declares victory with y at some round along (σ, τ) for some σ with first move x_0.
- Here, the statement "Player II declares victory" does not necessarily mean "Player II wins".
- Indeed, the above definition is made before f is specified, so the statement "Player II wins" does not make any sense.
- Again, one can remove x_0 from an input for $g^{\circlearrowleft}$ by considering a continuous strategy. The following is obvious by definition.

Observation

$f \leq_{\mathrm{W}} g^{\circlearrowleft} \iff f \leq_{\circlearrowleft\mathrm{W}} g$.

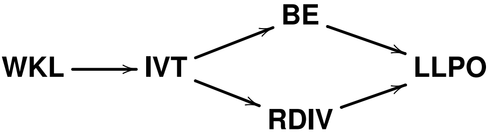
Westrick (2020) showed that for any partial multifunction f, note that $f^{\ni}$ is closed under the compositional product. Using this fact, one can show the following:

Theorem

Let $g :\subseteq \omega^\omega \to \omega^\omega$ be a partial multifunction.

Then $g^{\ni}$ is an idempotent jump operator on the Kleene-Vesley algebra $\mathbb{P}_{KV}$ such that the naïve $(\mathbb{P}_{KV}, g^{\ni})$-realizable partial multifunctions coincide with the partial multifunctions $\leq_{\ni W} g$.

- **LLPO**$^{\ni}$-realizability = Lifschitz' realizability in 1979.
- **WKL**$^{\ni}$-realizability = van Oosten's realizability in 1990.
- Let's consider **BE**$^{\ni}$-, **RDIV**$^{\ni}$- and **IVT**$^{\ni}$-realizability.



Theorem

1. RDIV $\not\leq^{c}_{\ni \mathrm{W}}$ BE.
2. BE $\not\leq^{c}_{\ni \mathrm{W}}$ RDIV.
3. IVT $\not\leq^{c}_{\ni \mathrm{W}}$ RDIV $\times$ BE.

Theorem

1. IZF + LLPO + ¬BE + ¬RDIV is LLPO$^{\ni}$-realizable.
2. IZF + BE + ¬RDIV is BE$^{\ni}$-realizable.
3. IZF + RDIV + ¬BE is RDIV$^{\ni}$-realizable.
4. IZF + BE + RDIV + ¬IVT is (BE + RDIV)$^{\ni}$-realizable.

McCarty Realizability

- First, we consider a set-theoretic universe with urlements $\mathbb{N}$.
- For a relative pca $\mathbb{P} = (P, \underset{\sim}{P})$, as in the usual set-theoretic forcing argument, we consider a *$\mathbb{P}$-name*, which is any set x satisfying the following condition:

$$x \subseteq \{(p, u) : p \in \underset{\sim}{P} \text{ and } (u \in \mathbb{N} \text{ or } u \text{ is a } \mathbb{P}\text{-name})\}.$$

- The $\mathbb{P}$-names are used as our universe. This notion can also be defined as the cumulative hierarchy:

$$V_0^{\mathbb{P}} = \emptyset, \quad V_\alpha^{\mathbb{P}} = \bigcup_{\beta<\alpha} \mathcal{P}(\underset{\sim}{P} \times (V_\beta^{\mathbb{P}} \cup \mathbb{N})), \text{ and } V_{\mathrm{set}}^{\mathbb{P}} = \bigcup_{\alpha \in \mathrm{Ord}} V_\alpha^{\mathbb{P}}.$$

- Note that the urelements $\mathbb{N}$ are disjoint from $V_{\mathrm{set}}^{\mathbb{P}}$.
  We define $V^{\mathbb{P}} = V_{\mathrm{set}}^{\mathbb{P}} \cup \mathbb{N}$.

Lifschitz-like Realizability for Set Theory

Chen-Rathjen (2012) introduced Lifschitz realizability for IZF.

We generalize Chen-Rathjen's realizability to **any** partial multifunction $\mathbf{j}$.

- Fix a partial multifunction $\mathbf{j} :\subseteq \underset{\sim}{\mathbf{P}} \rightrightarrows \underset{\sim}{\mathbf{P}}$. For $e \in \underset{\sim}{\mathbf{P}}$ and a sentence φ of $\mathbf{IZF}$ from parameters from $V^{\mathbb{P}}$, we define a relation $e \Vdash_{\mathbb{P}} \varphi$.
- For a primitive recursive relation R:

$$e \Vdash_{\mathbb{P}} R(\bar{a}) \iff \mathbb{N} \models R(\bar{a})$$
$$e \Vdash_{\mathbb{P}} \mathrm{N}(a) \iff a \in \mathbb{N} \ \ \& \ \ e = \underline{a}$$
$$e \Vdash_{\mathbb{P}} \mathrm{Set}(a) \iff a \in V^{\mathbb{P}}_{\mathrm{set}}$$

- For set-theoretic symbols

$$e \Vdash_{\mathbb{P}} a \in b \iff \forall^+ d \in \mathbf{j}(e) \exists c \, [(\pi_0 d, c) \in b \ \wedge \ \pi_1 d \Vdash_{\mathbb{P}} a = c]$$
$$\begin{aligned} e \Vdash_{\mathbb{P}} a = b \iff & (a, b \in \mathbb{N} \ \wedge \ a = b) \vee (\mathrm{Set}(a) \ \wedge \ \mathrm{Set}(b) \wedge \\ & \forall^+ d \in \mathbf{j}(e) \forall p, c \, [(p, c) \in a \ \rightarrow \ \pi_0 dp \Vdash_{\mathbb{P}} c \in b] \ \wedge \\ & \forall^+ d \in \mathbf{j}(e) \forall p, c \, [(p, c) \in b \ \rightarrow \ \pi_1 dp \Vdash_{\mathbb{P}} c \in a]. \end{aligned}$$

Here, we write $\forall^+ x \in X \, A(x)$ if both $X \neq \emptyset$ and $\forall x \in X \, A(x)$ hold.

- For logical connectives:

$$
\begin{aligned}
e \Vdash_{\mathbb{P}} A \wedge B &\iff \pi_0 e \Vdash_{\mathbb{P}} A \;\wedge\; \pi_1 e \Vdash_{\mathbb{P}} B \\
e \Vdash_{\mathbb{P}} A \vee B &\iff \forall^+ d \in \mathbf{j}(e)\, [(\pi_0 d = 0 \wedge \pi_1 d \Vdash_{\mathbb{P}} A) \\
&\qquad\qquad \vee (\pi_0 d = 1 \wedge \pi_1 d \Vdash_{\mathbb{P}} B)] \\
e \Vdash_{\mathbb{P}} \neg A &\iff (\forall a \in \underset{\sim}{P})\, a \nVdash_{\mathbb{P}} A \\
e \Vdash_{\mathbb{P}} A \rightarrow B &\iff (\forall a \in \underset{\sim}{P})\, [a \Vdash A \;\rightarrow\; ea \Vdash_{\mathbb{P}} B].
\end{aligned}
$$

- For quantifiers:

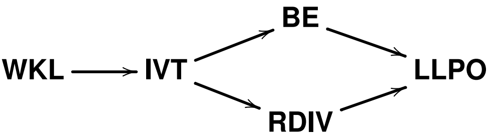
$$
\begin{aligned}
e \Vdash_{\mathbb{P}} \forall x A &\iff (\forall^+ d \in \mathbf{j}(e))(\forall c \in V^{\mathbb{P}})\, e \Vdash_{\mathbb{P}} A[c/x] \\
e \Vdash_{\mathbb{P}} \exists x A &\iff (\forall^+ d \in \mathbf{j}(e))(\exists c \in V^{\mathbb{P}})\, e \Vdash_{\mathbb{P}} A[c/x].
\end{aligned}
$$

- A formula φ is $\mathbf{j}$-*realizable over* $\mathbb{P}$ if there is $e \in P$ such that $e \Vdash_{\mathbb{P}} \varphi$.

Theorem

If j is an idempotent jump operator on $\mathbb{P}$,
then all axioms of **IZF** are j-realizable over $\mathbb{P}_j$.

Main Theorem



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3. IZF + **RDIV** + ¬**BE** is **RDIV**$^{\ni}$-realizable.
4. IZF + **BE** + **RDIV** + ¬**IVT** is (**BE** + **RDIV**)$^{\ni}$-realizable.