Cut-elimination in cyclic proof system for first-order logic

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What have we done?

- We find a counterexample to cut-elimination in $\mathtt{CLKID}^\omega$ i.e. the cyclic proof system for first-order logic with inductive definitions
- As noted by Brotherston [Brotherston, 2006]:
  > "***Conjecture 5.2.4.*** *Cut is not eliminable in the system* $CLKID^\omega$. *That is to say, there exists a sequent* $\Gamma \vdash \Delta$ *of* FOL_{ID} *which is provable in* $CLKID^\omega$, *but is not provable without the use of the rule (Cut).*"
- We show this conjecture is correct

Cyclic proof system

- Cyclic (circular) proof system
  - Each proof figure is usually a tree with cycles

    [Brotherston, 2006, Brotherston and Simpson, 2011]
  - Examples of application for software verification
    - Verifying properties of concurrent processes [Schöpp and Simpson, 2002]
    - Termination of pointer programs [Brotherston et al., 2008]
    - Decision procedure for symbolic heaps

      [Brotherston et al., 2012, Chu et al., 2015, Ta et al., 2017, Tatsuta et al., 2019]

Related works

- There exist the cut-free and complete cyclic proof systems for modal μ-calculus [Stirling, 2014, Afshari and Leigh, 2017], linear time μ-calculus [Brünnler and Lange, 2008] and Gödel-Löb provability logic [Shamkanov, 2014]
- Kimura et al. [KIMURA et al., 2020] show that the cut-elimination property in the cyclic proof system for separation logic does not hold

Inductive predicate 1

Inductive predicate: a formalisation of the inductive definition with productions

Definition 1 (Inductive definition set)

A production is defined as

$$\frac{Q_1\mathbf{u}_1 \quad \cdots \quad Q_h\mathbf{u}_h \quad P_{j_1}\mathbf{t}_1 \quad \cdots \quad P_{j_m}\mathbf{t}_m}{P_i\mathbf{t}} \quad ,$$

where each of $Q_1\mathbf{u}_1, \ldots, Q_h\mathbf{u}_h$ denotes an atomic formula with an ordinal predicate symbol and each of $P_{j_1}\mathbf{t}_1, \ldots, P_{j_m}\mathbf{t}_m$ and $P_i\mathbf{t}$ denotes an atomic formula with an inductive predicate symbol.

Inductive predicate 2

Examples:

Natural Number $\mathrm{N}(x)$

$$\frac{}{\mathrm{N}(0)} \qquad \frac{\mathrm{N}(x)}{\mathrm{N}(sx)}$$

Addition $\mathrm{Add}_1(x, y, z)$ $\quad x + y = z$

$$\frac{}{\mathrm{Add}_1(0, y, y)} \qquad \frac{\mathrm{Add}_1(x, y, z)}{\mathrm{Add}_1(sx, y, sz)}$$

Inductive predicate 3

Note

- An inductive definition set is a finite set of productions.
- The semantics of inductive predicates is obtained by considering the least fixed point of a monotone operator constructed from the inductive definition set [Brotherston and Simpson, 2011].

Inference rules of $\mathtt{CLKID}^\omega$

Each red formula is the principal formula of each rule

$$\Gamma \cap \Delta = \emptyset \; \frac{}{\Gamma \vdash \Delta}\,(\text{Axiom}) \qquad \Gamma' \subset \Gamma,\ \Delta' \subset \Delta \; \frac{\Gamma' \vdash \Delta'}{\Gamma \vdash \Delta}\,(\text{Weak}) \qquad \frac{\Gamma \vdash F, \Delta \qquad \Gamma, F \vdash \Delta}{\Gamma \vdash \Delta}\,(\text{Cut}) \qquad \frac{\Gamma \vdash \Delta}{\Gamma[\theta] \vdash \Delta[\theta]}\,(\text{Subst})$$

$$\frac{\Gamma \vdash F, \Delta}{\Gamma, \neg F \vdash \Delta}\,(\neg\ \text{L}) \qquad \frac{\Gamma, F \vdash \Delta}{\Gamma \vdash \neg F, \Delta}\,(\neg\ \text{R}) \qquad \frac{\Gamma, F \vdash \Delta \qquad \Gamma, G \vdash \Delta}{\Gamma, F \vee G \vdash \Delta}\,(\vee\ \text{L}) \qquad \frac{\Gamma \vdash F, G, \Delta}{\Gamma \vdash F \vee G, \Delta}\,(\vee\ \text{R})$$

$$\frac{\Gamma, F, G \vdash \Delta}{\Gamma, F \wedge G \vdash \Delta}\,(\wedge\ \text{L}) \qquad \frac{\Gamma \vdash F, \Delta \qquad \Gamma \vdash G, \Delta}{\Gamma \vdash F \wedge G, \Delta}\,(\wedge\ \text{R}) \qquad \frac{\Gamma \vdash F, \Delta \qquad \Gamma, G \vdash \Delta}{\Gamma, F \to G \vdash \Delta}\,(\to\ \text{L}) \qquad \frac{\Gamma, F \vdash G, \Delta}{\Gamma \vdash F \to G, \Delta}\,(\to\ \text{R})$$

$$\frac{\Gamma, F[x := t] \vdash \Delta}{\Gamma, \forall x F \vdash \Delta}\,(\forall\ \text{L}) \qquad x \notin \text{FV}(\Gamma \cup \Delta)\; \frac{\Gamma \vdash F, \Delta}{\Gamma \vdash \forall x F, \Delta}\,(\forall\ \text{R}) \qquad x \notin \text{FV}(\Gamma \cup \Delta)\; \frac{\Gamma, F \vdash \Delta}{\Gamma, \exists x F \vdash \Delta}\,(\exists\ \text{L}) \qquad \frac{\Gamma \vdash F[x := t], \Delta}{\Gamma \vdash \exists x F, \Delta}\,(\exists\ \text{R})$$

$$\frac{\Gamma[x := u, y := t] \vdash \Delta[x := u, y := t]}{\Gamma[x := t, y := u], t = u \vdash \Delta[x := t, y := u]}\,(=\ \text{L}) \qquad \frac{}{\Gamma \vdash t = t, \Delta}\,(=\ \text{R})$$

The inference rules for inductive predicates 1

For each production

$$\frac{Q_1\mathbf{u}_1(\mathbf{x}) \quad \cdots \quad Q_h\mathbf{u}_h(\mathbf{x}) \quad P_{j_1}\mathbf{t}_1(\mathbf{x}) \quad \cdots \quad P_{j_m}\mathbf{t}_m(\mathbf{x})}{P_i\mathbf{t}(\mathbf{x})} \, ,$$

there is a corresponding inference rule

$$\frac{\Gamma \vdash Q_1\mathbf{u}_1(\mathbf{u}), \Delta \;\cdots\; \Gamma \vdash Q_h\mathbf{u}_h(\mathbf{u}), \Delta \;\; \Gamma \vdash P_{j_1}\mathbf{t}_1(\mathbf{u}), \Delta \;\cdots\; \Gamma \vdash P_m\mathbf{t}_m(\mathbf{u}), \Delta}{\Gamma \vdash P_i\mathbf{t}(\mathbf{u}), \Delta} (P_i \, \mathrm{R}_r) \; .$$

The inference rules for inductive predicates 2

The left introduction rule for inductive predicates is

$$\frac{\text{case distinctions}}{\Gamma, P_i \mathbf{u} \vdash \Delta}\ (\mathrm{Case}\ P_i) \quad ,$$

where for each production having the predicate P_i in its conclusion

$$\frac{Q_1 \mathbf{u}_1(\mathbf{x}) \quad \ldots \quad Q_h \mathbf{u}_h(\mathbf{x}) \quad P_{j_1} \mathbf{t}_1(\mathbf{x}) \quad \ldots \quad P_{j_m} \mathbf{t}_m(\mathbf{x})}{P_i \mathbf{t}(\mathbf{x})} \quad ,$$

there is a corresponding case distinction

$$\Gamma, \mathbf{u} = \mathbf{t}(\mathbf{y}), Q_1 \mathbf{u}_1(\mathbf{y}), \ldots, Q_h \mathbf{u}_h(\mathbf{y}), P_{j_1} \mathbf{t}_1(\mathbf{y}), \ldots, P_{j_m} \mathbf{t}_m(\mathbf{y}) \vdash \Delta,$$

where $\mathbf{y}$ is a sequence of distinct fresh variables of the same length as $\mathbf{x}$.

The inference rules for Add_1

$$\frac{}{\mathrm{Add}_1(0, y, y)} \qquad \frac{\mathrm{Add}_1(x, y, z)}{\mathrm{Add}_1(sx, y, sz)}$$

$$\frac{}{\Gamma \vdash \Delta, \mathrm{Add}_1(0, b, b)}\,(\mathrm{Add}_1\ \mathrm{R}_1) \qquad \frac{\Gamma \vdash \Delta, \mathrm{Add}_1(a, b, c)}{\Gamma \vdash \Delta, \mathrm{Add}_1(sa, b, sc)}\,(\mathrm{Add}_1\ \mathrm{R}_2)$$

$$\frac{\Gamma, a = 0, b = y, c = y \vdash \Delta \qquad \Gamma, a = sx, b = y, c = sz, \mathrm{Add}_1(x, y, z) \vdash \Delta}{\Gamma, \mathrm{Add}_1(a, b, c) \vdash \Delta}\,(\mathrm{Case}\ \mathrm{Add}_1)\quad ,$$

where x, y, z are fresh

The inference rules for inductive predicates 3

Note

Case-descendants of the principal formula $P_i\mathbf{u}$ in a $(\mathrm{Case}\ P_i)$ are the formulas

$P_{j_1}\mathbf{t}_1(\mathbf{y}), \ldots, P_{j_m}\mathbf{t}_m(\mathbf{y})$.

Trace 1

T : a derivation figure

$(\Gamma_i \vdash \Delta_i)_{0 \le i < \alpha}$: a path in T

Define a trace following $(\Gamma_i \vdash \Delta_i)_{0 \le i < \alpha}$ to be a sequence of formulas $(\tau_i)_{0 \le i < \alpha}$ such that the following hold:

1. Each τ_i is an element of Γ_i and an atomic formula with an inductive predicate.
2. If $\Gamma_i \vdash \Delta_i$ is the conclusion of (Subst), then $\tau_i \equiv \tau_{i+1}[\theta]$, where θ is the substitution associated with the rule instance.

Trace 2

3. If $\Gamma_i \vdash \Delta_i$ is the conclusion of $(= \mathrm{L})$ with the principal formula $t = u$, then there are a formula F and variables x, y such that $\tau_i \equiv F[x := t, y := u]$ and $\tau_{i+1} \equiv F[x := u, y := t]$.
4. If $\Gamma_i \vdash \Delta_i$ is the conclusion of $(\mathrm{Case}\ P_i)$, then either
   - τ_i is the principal formula of the rule instance and τ_{i+1} is a case-descendant of τ_i or
   - τ_{i+1} is the same as τ_i.

   In the former case, i is said to be a progress point of the trace.

Trace 3

5. If $\Gamma_i \vdash \Delta_i$ is the conclusion of any other rules, then $\tau_{i+1} \equiv \tau_i$.

Note

Call a trace having infinitely many progress points an infinitely progressing trace.

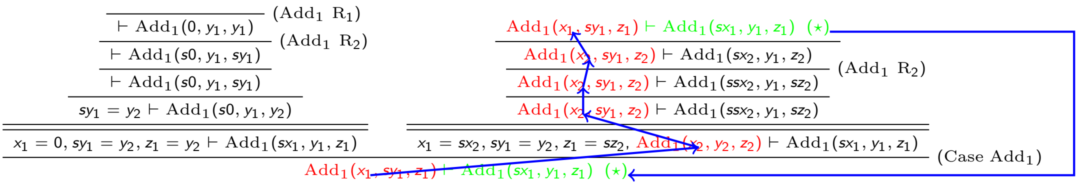
Global trace condition

Definition 2 (Global trace condition)

If, for every infinite path $(v_i)_{i \geq 0}$ in a derivation figure, there exists an infinitely progressing trace following a tail of the path $(v_i)_{i \geq k}$ with $k \geq 0$, then we say the derivation figure satisfies the global trace condition.

A $\mathtt{CLKID}^{\omega}$ proof is a derivation tree with cycles satisfying the global trace condition.

Examples of $\texttt{CLKID}^\omega$ proof



Red formulas denote an infinitely many progressing trace

Green sequents denote a bud and the corresponding companion

The blue line denotes an infinite path

The inference rules for Add_2

$\mathrm{Add}_2(x, y, z) : x + y = z$

$$\frac{}{\mathrm{Add}_2(0, y, y)} \qquad \frac{\mathrm{Add}_2(x, sy, z)}{\mathrm{Add}_2(sx, y, z)}$$

$$\frac{}{\vdash \mathrm{Add}_2(0, b, b)}\ (\mathrm{Add}_2\ \mathrm{R}_1) \qquad \frac{\Gamma \vdash \Delta, \mathrm{Add}_2(a, sb, c)}{\Gamma \vdash \Delta, \mathrm{Add}_2(sa, b, c)}\ (\mathrm{Add}_2\ \mathrm{R}_2)$$

$$\frac{\Gamma, a = 0, b = y, c = y \vdash \Delta \qquad \Gamma, a = sx, b = y, c = z, \mathrm{Add}_2(x, sy, z) \vdash \Delta}{\Gamma, \mathrm{Add}_2(a, b, c) \vdash \Delta}\ (\mathrm{Case}\ \mathrm{Add}_2) \quad ,$$

where x, y, z are fresh

Main theorem

Theorem 3 (Main theorem)

1. $\mathrm{Add}_2(x, y, z) \vdash \mathrm{Add}_1(x, y, z)$ *is provable in* $\mathtt{CLKID}^{\omega}$*.*
2. $\mathrm{Add}_2(x, y, z) \vdash \mathrm{Add}_1(x, y, z)$ *is not cut-free provable in* $\mathtt{CLKID}^{\omega}$*.*

$\mathtt{CLKID}^{\omega}$ proof of $\mathrm{Add}_2(x, y, z) \vdash \mathrm{Add}_1(x, y, z)$

$\mathcal{D}_1$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash \mathrm{Add}_1(0, y_1, y_1)}\ (\mathrm{Add}_1\ \mathrm{R}_1)}{\vdash \mathrm{Add}_1(s0, y_1, sy_1)}\ (\mathrm{Add}_1\ \mathrm{R}_2)}{\vdash \mathrm{Add}_1(s0, y_1, sy_1)}}{sy_1 = y_2 \vdash \mathrm{Add}_1(s0, y_1, y_2)}}{x_1 = 0, sy_1 = y_2, z_1 = y_2 \vdash \mathrm{Add}_1(sx_1, y_1, z_1)}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\mathrm{Add}_1(x_1, sy_1, z_1) \vdash \mathrm{Add}_1(sx_1, y_1, z_1)\ (\star)}{\mathrm{Add}_1(x_2, sy_1, z_2) \vdash \mathrm{Add}_1(sx_2, y_1, z_2)}}{\mathrm{Add}_1(x_2, sy_1, z_2) \vdash \mathrm{Add}_1(ssx_2, y_1, sz_2)}\ (\mathrm{Add}_1\ \mathrm{R}_2)}{\mathrm{Add}_1(x_2, sy_1, z_2) \vdash \mathrm{Add}_1(ssx_2, y_1, sz_2)}}{x_1 = sx_2, sy_1 = y_2, z_1 = sz_2, \mathrm{Add}_1(x_2, y_2, z_2) \vdash \mathrm{Add}_1(sx_1, y_1, z_1)}$$

$$\dfrac{\cdots \qquad \cdots}{\mathrm{Add}_1(x_1, sy_1, z_1) \vdash \mathrm{Add}_1(sx_1, y_1, z_1)\ (\star)}\ (\mathrm{Case}\ \mathrm{Add}_1)$$

$\mathcal{D}_2$

$$\dfrac{\dfrac{}{\vdash \mathrm{Add}_1(0, y_1, y_1)}\ (\mathrm{Add}_1\ \mathrm{R}_1)}{x = 0, y = y_1, z = y_1 \vdash \mathrm{Add}_1(x, y, z)}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\mathrm{Add}_2(x, y, z) \vdash \mathrm{Add}_1(x, y, z)\ (\dagger)}{\mathrm{Add}_2(x_1, sy_1, z_1) \vdash \mathrm{Add}_1(x_1, sy_1, z_1)} \qquad \overset{\mathcal{D}_1}{\mathrm{Add}_1(x_1, sy_1, z_1) \vdash \mathrm{Add}_1(sx_1, y_1, z_1)}}{\mathrm{Add}_2(x_1, sy_1, z_1) \vdash \mathrm{Add}_1(sx_1, y_1, z_1)}\ (\mathrm{Cut})}{x = sx_1, y = y_1, z = z_1, \mathrm{Add}_2(x_1, sy_1, z_1) \vdash \mathrm{Add}_1(x, y, z)}}{}$$

$$\dfrac{\cdots \qquad \cdots}{\mathrm{Add}_2(x, y, z) \vdash \mathrm{Add}_1(x, y, z)\ (\dagger)}\ (\mathrm{Case}\ \mathrm{Add}_2)$$

Proof sketch of the main theorem 1

- Assume there exists a cut-free $\mathtt{CLKID}^{\omega}$ proof $\mathcal{P}$ of $\mathrm{Add}_2(x, y, z) \vdash \mathrm{Add}_1(x, y, z)$

Definition 4 ($\cong_\Gamma$)

Γ : a set of formulas in which all terms occurring are of the form $s^n x$ or $s^n 0$.

Define the relation $\cong_\Gamma$ to be the smallest congruence relation on terms which satisfies that $t_1 = t_2 \in \Gamma$ implies $t_1 \cong_\Gamma t_2$.

Proof sketch of the main theorem 2

Definition 5 (Index)

$\Gamma \vdash \Delta$: a sequent Fix $\mathrm{Add}_2(a, b, c) \in \Gamma$.

Define the index of $\mathrm{Add}_2(a, b, c)$ in $\Gamma \vdash \Delta$ as follows:

1. If $s^n b \not\cong_\Gamma s^m b'$ for any $n, m \in \mathbb{N}$, $\mathrm{Add}_1(a', b', c') \in \Delta$, then the index is $\perp$.
2. If $\exists n, \exists m \in \mathbb{N}$ s.t. $s^n b \cong_\Gamma s^m b'$ for some $\mathrm{Add}_1(a', b', c') \in \Delta$ and $m - n$ is unique (namely $s^{n'} b \cong_\Gamma s^{m'} b'$ implies $m - n = m' - n'$ for any $\mathrm{Add}_1(a', b', c') \in \Delta$), then the index is $m - n$.

Proof sketch of the main theorem 3

Definition 6 (Index sequent)

The sequent $\Gamma \vdash \Delta$ is said to be an index sequent if the following conditions hold:

1. If $t \in \mathrm{B}_1(\Gamma \vdash \Delta)$ and $u \in \mathrm{C}(\Gamma \vdash \Delta)$, then $s^n t \not\cong_\Gamma s^m u$ for any $n, m \in \mathbb{N}$.
2. If $s^n b \cong_\Gamma s^m b'$ with $b, b' \in \mathrm{B}_1(\Gamma \vdash \Delta)$, then $n = m$.

$$\mathrm{B}_1(\Gamma \vdash \Delta) = \{b | \mathrm{Add}_1(a, b, c) \in \Delta\},$$

$$\mathrm{C}(\Gamma \vdash \Delta) = \{c | \mathrm{Add}_2(a, b, c) \in \Gamma \text{ or } \mathrm{Add}_1(a, b, c) \in \Delta\}.$$

Proof sketch of the main theorem 4

Lemma 7

The index of each $\mathrm{Add}_2(a, b, c)$ *in an index sequent is defined.*

Proof sketch of the main theorem 5

A switching point is a node where the corresponding rule is $(\mathrm{Case\ Add}_2)$ whose principal formula's index is $\perp$.

$$
\dfrac{\begin{matrix}\vdots \\ \end{matrix} \quad \dfrac{\begin{matrix}\vdots\end{matrix} \quad \dfrac{\dfrac{(\heartsuit)\ \mathrm{Add}_2(x_1, sy_1, z_1)[\perp], \mathrm{Add}_2(x, y, z)[0] \vdash \mathrm{Add}_1(x, y, z)}{\mathrm{Add}_2(x_2, sy_2, z_2)[\perp], \mathrm{Add}_2(x, y, z)[0] \vdash \mathrm{Add}_1(x, y, z)}}{x_1 = sx_2, sy_1 = y_2, z_1 = z_2, \mathrm{Add}_2(x_2, sy_2, z_2)[\perp], \mathrm{Add}_2(x, y, z)[0] \vdash \mathrm{Add}_1(x, y, z)}}{\dfrac{(\heartsuit)\ \mathrm{Add}_2(x_1, sy_1, z_1)[\perp], \mathrm{Add}_2(x, y, z)[0] \vdash \mathrm{Add}_1(x, y, z)}{x = sx_1, y = y_1, z = z_1, \mathrm{Add}_2(x_1, sy_1, z_1)[1], \mathrm{Add}_2(x, y, z)[0] \vdash \mathrm{Add}_1(x, y, z)}} \ (\mathrm{Switching})}{\mathrm{Add}_2(x, y, z)[0] \vdash \mathrm{Add}_1(x, y, z)} \ (\mathrm{Case\ Add}_2)
$$

Proof sketch of the main theorem 6

Definition 8 (Index path)

A path $(v_i)_{0 \leq i < \alpha}$ is said to be an index path if the following conditions hold:

1. $\mathrm{Seq}(v_0)$ is an index sequent
2. If v_i is the left assumption of $(\mathrm{Case\ Add}_2)$ for $i > 0$, v_{i-1} is a switching point

Note

The rightmost path from an index path is an index path.

Proof sketch of the main theorem 7

Lemma 9

Every sequent in an index path is an index sequent.

Lemma 10

Let $(v_i)_{i \geq 0}$ *be an infinite index path.*

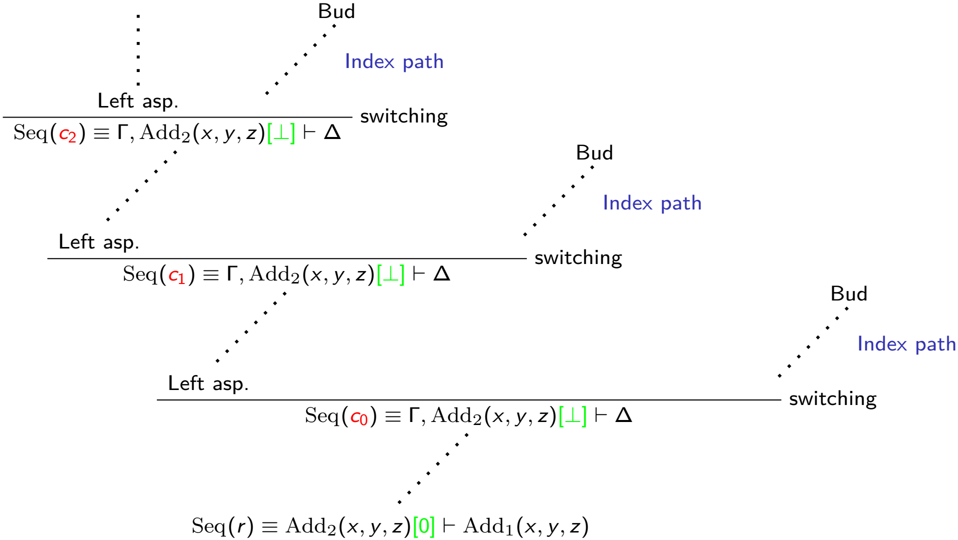
There exists $l \in \mathbb{N}$ *such that the following conditions satisfy:*

1. $\mathrm{Rule}(v_l)$ *is a switching point*
2. v_{l+1} *is the right assumption of* $(Case\ \mathrm{Add}_2)$

Proof sketch of the main theorem 8

Show that there exists a sequence $(c_i)_{i \in \mathbb{N}}$ of nodes in $\mathcal{P}$ which satisfies the following conditions:

1. c_i is a switching point.
2. For a node v on the path from the root to c_n with $\mathrm{Rule}(v) = (\mathrm{Case\ Add}_2)$ for each $n \in \mathbb{N}$, v is a switching point if and only if the child of v on the path from the root to c_n is the left assumption of v.
3. For all $i, j \in \mathbb{N}$ with $j > i$, the height of c_j is greater than the height of c_i.



By repeating this process, we get a set of infinite nodes $\{c_i | i \in \mathbb{N}\}$. CONTRADICTION!!

Conclusion

Main result

There exists a sequent, not cut-free provable in $\texttt{CLKID}^{\omega}$ but provable in $\texttt{CLKID}^{\omega}$.

In other words, (Cut) cannot be eliminated in $\texttt{CLKID}^{\omega}$.

Future work

Can we restrict principal formulas of (Cut) without changing provability?

Reference 1

[Afshari and Leigh, 2017] Afshari, B. and Leigh, G. E. (2017).

Cut-free completeness for modal mu-calculus.

In *2017 32nd Annual ACM/IEEE Symposium on Logic in Computer Science (LICS)*, pages 1–12.

[Brotherston, 2006] Brotherston, J. (2006).

Sequent Calculus Proof Systems for Inductive Definitions.

PhD thesis, University of Edinburgh.

[Brotherston et al., 2008] Brotherston, J., Bornat, R., and Calcagno, C. (2008).

Cyclic proofs of program termination in separation logic.

SIGPLAN Not., 43(1):101–112.

Reference 2

[Brotherston et al., 2012] Brotherston, J., Gorogiannis, N., and Petersen, R. L. (2012).

A Generic Cyclic Theorem Prover.

In Hutchison, D., Kanade, T., Kittler, J., Kleinberg, J. M., Mattern, F., Mitchell, J. C., Naor, M., Nierstrasz, O., Pandu Rangan, C., Steffen, B., Sudan, M., Terzopoulos, D., Tygar, D., Vardi, M. Y., Weikum, G., Jhala, R., and Igarashi, A., editors, *Programming Languages and Systems*, volume 7705, pages 350–367. Springer Berlin Heidelberg, Berlin, Heidelberg.

[Brotherston and Simpson, 2011] Brotherston, J. and Simpson, A. (2011).

Sequent calculi for induction and infinite descent.

Journal of Logic and Computation, 21(6):1177–1216.

Reference 3

[Brünnler and Lange, 2008] Brünnler, K. and Lange, M. (2008).

Cut-free sequent systems for temporal logic.

The Journal of Logic and Algebraic Programming, 76(2):216 – 225.

[Chu et al., 2015] Chu, D.-H., Jaffar, J., and Trinh, M.-T. (2015).

Automatic induction proofs of data-structures in imperative programs.

In *Proceedings of the 36th ACM SIGPLAN Conference on Programming Language Design and Implementation*, pages 457–466.

[KIMURA et al., 2020] KIMURA, D., NAKAZAWA, K., TERAUCHI, T., and UNNO, H. (2020).

Failure of cut-elimination in cyclic proofs of separation logic.

Computer Software, 37(1):39–52.

Reference 4

[Saotome et al., 2020] Saotome, K., Nakazawa, K., and Kimura, D. (2020).

Restriction on Cut in Cyclic Proof System for Symbolic Heaps.

In Nakano, K. and Sagonas, K., editors, *Functional and Logic Programming*, pages 88–105. Springer International Publishing.

[Schöpp and Simpson, 2002] Schöpp, U. and Simpson, A. (2002).

Verifying Temporal Properties Using Explicit Approximants: Completeness for Context-free Processes.

In Nielsen, M. and Engberg, U., editors, *Foundations of Software Science and Computation Structures*, pages 372–386. Springer Berlin Heidelberg.

[Shamkanov, 2014] Shamkanov, D. S. (2014).

Circular proofs for the gödel-löb provability logic.

Mathematical Notes, 96(3-4):575–585.

Reference 5

[Stirling, 2014] Stirling, C. (2014).

A tableau proof system with names for modal mu-calculus.

In Voronkov, A. and Korovina, M., editors, *HOWARD-60. A Festschrift on the Occasion of Howard Barringer's 60th Birthday*, volume 42 of *EPiC Series in Computing*, pages 306–318. EasyChair.

[Ta et al., 2017] Ta, Q.-T., Le, T. C., Khoo, S.-C., and Chin, W.-N. (2017).

Automated lemma synthesis in symbolic-heap separation logic.

Proceedings of the ACM on Programming Languages, 2(POPL):1–29.

[Tatsuta et al., 2019] Tatsuta, M., Nakazawa, K., and Kimura, D. (2019).

Completeness of Cyclic Proofs for Symbolic Heaps with Inductive Definitions.

In Lin, A. W., editor, *Programming Languages and Systems*, pages 367–387. Springer International Publishing.