Decidability of variables in constructive logics

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22/Mar/2021

Introduction

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- We shall see how we can refine Ishii's class by using weaker principles than decidability.

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- (Ishihara 2014) and (Ishii 2018) proposed two incomparable classes.
- We shall see how we can refine Ishii's class by using weaker principles than decidability.
- This will also allow us to extend the result to weaker logics.

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Preliminary

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Refining Ishii's class

Extension to minimal logic

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$(A \wedge B) \to A$; $(A \wedge B) \to B$; $A \to (B \to A \wedge B)$;
$A \to (A \vee B)$; $B \to (A \vee B)$;
$(A \to C) \to ((B \to C) \to ((A \vee B) \to C))$;
$A \vee \neg A$ [LEM];
$\bot \to A$ [EFQ].

$$\frac{A \to B \qquad A}{B}\ (\mathrm{MP})$$

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$$\frac{A \to B \qquad A}{B} \text{ (MP)}$$

We write $\Gamma \vdash_c A$ for the derivability in **CPC**.

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$$\frac{A, B, \Gamma \Rightarrow \Delta}{A \wedge B, \Gamma \Rightarrow \Delta} \text{ (L}\wedge\text{)} \qquad \frac{\Gamma \Rightarrow \Delta, A \qquad \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \wedge B} \text{ (R}\wedge\text{)}$$

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A solution to this question implies the conservativity of a classical consequence to **IPC**, if V turns out to be empty for some Γ and A.

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\mathcal{V}^+(\bot) &= \emptyset & \mathcal{V}^-(\bot) &= \emptyset \\
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For a set of formulae Γ, $\mathcal{V}^+(\Gamma)$ and $\mathcal{V}^-(\Gamma)$ are similarly defined.

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- He showed If $\vdash_{3c} \Gamma, \Delta \Rightarrow \Sigma$, then $\vdash_{3i} \Pi_V, \Gamma, \neg\Delta \to *, \Sigma \to * \Rightarrow *$. for a place-holder $*$.

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- Then if $\Sigma = \{A\}$, substitute $*$ by A to obtain $\vdash_{3i} \Pi_V, \Gamma \Rightarrow A$ (with $\Delta = \emptyset$).

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Theorem (Glivenko 1929)

If $\Gamma \vdash_c A$ then $\Gamma \vdash_i \neg\neg A$.

- Then it is a matter of finding V so that $\Pi_v \vdash_i \neg\neg A \to A$.

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Definition (Ishii 2018)

We define $\widetilde{\mathcal{E}}_A$ inductively.

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\widetilde{\mathcal{E}}_p &:= \{p \vee \neg p\} \\
\widetilde{\mathcal{E}}_\bot &:= \emptyset \\
\widetilde{\mathcal{E}}_{A \wedge B} &:= \widetilde{\mathcal{E}}_A \cup \widetilde{\mathcal{E}}_B \\
\widetilde{\mathcal{E}}_{A \vee B} &:= \widetilde{\mathcal{E}}_A \cup \mathcal{E}_B \text{ or } \mathcal{E}_A \cup \widetilde{\mathcal{E}}_B \\
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($\widetilde{\mathcal{E}}_A$ is therefore *non-deterministic*.)

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- Ishii observes that each of the solutions can perform better than the other, depending on $\Gamma \cup \{A\}$.
- Very roughly, If $\Gamma \vdash_c A$:
  - Ishihara's class: can drop strictly positive occurrences in Γ;
  - Ishii's class: only needs strictly positive occurrences in A (except for disjunctions).

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- But for this assuming $\neg\neg p \to p$ surely suffices.
- Hence there seems to be a room for improvement for Ishii's class.
- In particular, it appears promising to use a weaker principle than LEM.

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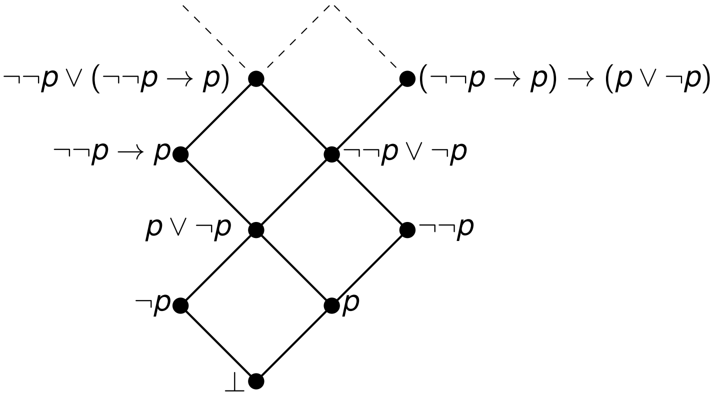
Recall *Rieger-Nishimura lattice* (Rieger 1949, Nishimura 1960), the Lindenbaum algebra of one-variable formulae.



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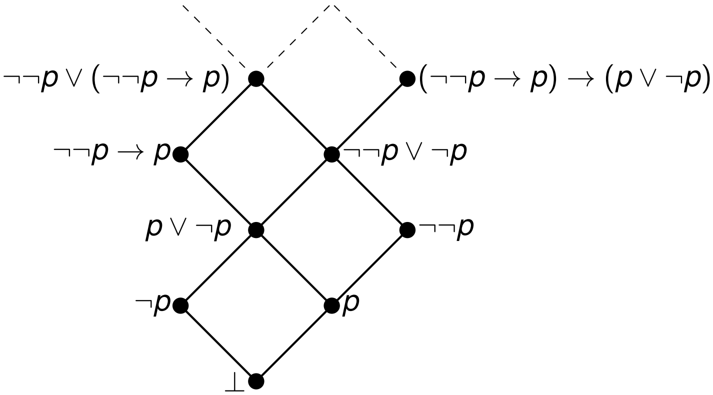

From this it seems reasonable to consider classes of $\neg\neg p \vee \neg p$ (WLEM) and $\neg\neg p \to p$ (DNE).

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- Can we then add $\neg\neg(\bot \to A)$ to **MPC** without making it **IPC**?
- The answer is in the affirmative.

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- Equivalently one can add $\neg A \to \neg\neg(A \to B)$.
- We shall call it **GPC** (derivability $\vdash_g$).
- it is the smallest extension of **MPC** with respect to which Glivenko's theorem holds.

A thing about WLEM

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- For this, we use AVQ to infer $\neg B \to \neg\neg(B \to C)$ for the case $A \equiv (B \to C)$.
- Note we cannot use LEM, because $\nvdash_g \neg B \to (B \to C)$.
- So extension of Ishii's method to Glivenko's logic *requires* us to think in terms of WLEM and DNE.

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\widetilde{\mathcal{W}}_{A \wedge B} &= \widetilde{\mathcal{W}}_A \cup \widetilde{\mathcal{W}}_B \\
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So we take an instance from one of the disjuncts for each disjunction occurring strictly positively.

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Let $*_1, *_2, \ldots$ be a countable set of symbols. The class $\mathcal{F}$ of *multiple formula contexts* is defined inductively as follows. (where $F, F' \in \mathcal{F}$ and A a formula.)

(i) $*_n, \perp, A \to F \in \mathcal{F}$.
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(i) $*_n, \bot, A \to F \in \mathcal{F}$.
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Note any formula can be written as $F[p_1, \dots, p_n]$.

Class of WLEM

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Proposition

Let $F[*_1, \ldots, *_n] \in \mathcal{F}$. Then
$\widetilde{\mathcal{W}}_{F[p_1,\ldots,p_n]} \vdash_g \neg\neg F[p_1, \ldots, p_n] \to F[\neg\neg p_1, \ldots, \neg\neg p_n]$.

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That is to say, we can push the double negations inside, to the front of strictly positive propositional variables.

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Theorem

If $\Gamma \vdash_c A$, then $\widetilde{\mathcal{W}}_A, \mathcal{D}_A, \Gamma \vdash_g A$.

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- Then $\neg\neg p \vee \neg p, \neg\neg q \to q \vdash_g \neg\neg(p \vee q) \to (\neg\neg p \vee q)$.
- With the same choice of disjuncts, Ishii's class gives $\{p \vee \neg p, q \vee \neg q\}$.
- For the other possible choice, the classes give $\{\neg\neg q \to q, \neg\neg q \vee \neg q\}$ and $\{q \vee \neg q\}$, respectively.

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- So our class always give at least as good, and sometimes strictly better, solutions compared to Ishii's.
- In addition, our approach enabled to treat Glivenko's logic as well.

Outline

Preliminary

Decidability of variables

Refining Ishii's class

Extension to minimal logic

Where did we rely on AVQ?

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- We relied on AVQ in two places.
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  2. In showing $\mathcal{W}_A \vdash_g \neg\neg A \vee \neg A$.
- We shall first see how to evade from the former reliance.

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Definition (Gödel-Gentzen translation)

For each formula A, We define its translation $()^g$ by the following clauses.

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\begin{aligned}
p^g &\equiv \neg\neg p \\
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(A \wedge B)^g &\equiv A^g \wedge B^g \\
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We shall write $\Gamma^g = \{A^g : A \in \Gamma\}$.

Theorem

(i) *For any A, $\vdash_m \neg\neg A^g \leftrightarrow A^g$.*
(ii) *If $\Gamma \vdash_c A$, then $\Gamma^g \vdash_m A^g$.*

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That is, $\widetilde{\mathcal{Q}}_A$ collects propositional variables occuring in the conclusions of implications.

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Given a formula A, we say it is *Q-spreading* if $\widetilde{\mathcal{Q}}_A \vdash_m A \to A^g$, and *Q-isolating* if $\widetilde{\mathcal{Q}}_A \vdash_m A^g \to \neg\neg A$.

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Any formula is both Q-spreading and Q-isolating.

Corollary

If $\Gamma \vdash_c A$, then $\widetilde{\mathcal{Q}}_{\Gamma \cup \{A\}}, \Gamma \vdash_m \neg\neg A$.

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- This means all propositional variables introduced by (L$\bot$) in a proof of **G3i** occurs in one of these positions.
- Hence it suffices to assume EFQ for such instances to preserve the derivation into **MPC**.
- In particular, for $\vdash_{3i} \Gamma \Rightarrow \neg\neg A$, it turns out that instances of AVQ are sufficient.

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Theorem

If $\Gamma \vdash_c A$, then $\mathcal{B}_{\Gamma \cup \{A\}}, \Gamma \vdash_m \neg\neg A$.

Proof.

If $\Gamma \vdash_c A$, then $\Gamma \vdash_i \neg\neg A$. So $\{\bot \to p : p \in \mathcal{V}^-(\Gamma) \cup \mathcal{V}^+(A)\}, \Gamma \vdash_m \neg\neg A$. Thus by contraposing multiple times, we obtain $\mathcal{B}_{\Gamma \cup \{A\}}, \Gamma \vdash_m \neg\neg A$. □

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- For $A \equiv (\bot \to p) \vee \neg\neg q$ we see $\widetilde{\mathcal{Q}}_A = \{p\}$ but $\mathcal{B}_A = \{p, q\}$.
- For $A \equiv \bot \to (q \to p)$ we have $\widetilde{\mathcal{Q}}_A = \{p, q\}$ but $\mathcal{B}_A = \{p\}$.
- Hence it depends on the formula which one of $\widetilde{\mathcal{Q}}_A$ and $\mathcal{B}_A$ gives a better result.

Last step

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- We have $\mathcal{Q}_C, \mathcal{W}_C \vdash_m \neg\neg C \vee \neg C$.
- So $\mathcal{Q}$ in addition to $\mathcal{W}$ suffices to enable our argument for **MPC**.

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Definition

We define $\widetilde{\mathcal{R}}_A$ inductively.

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\widetilde{\mathcal{R}}_{A\wedge B} &= \widetilde{\mathcal{R}}_A \cup \widetilde{\mathcal{R}}_B \\
\widetilde{\mathcal{R}}_{A\vee B} &= \widetilde{\mathcal{R}}_A \cup \mathcal{Q}_B \cup \mathcal{W}_B \text{ or } \mathcal{Q}_A \cup \mathcal{W}_A \cup \widetilde{\mathcal{R}}_B \\
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Proposition

(i) If $\Gamma \vdash_c A$, then $\widetilde{\mathcal{Q}}_{\Gamma \cup \{A\}}, \widetilde{\mathcal{R}}_A, \mathcal{D}_A, \Gamma \vdash_m A$.

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In particular, since $\vdash_m (\neg\neg p \to p) \to (\bot \to p)$, denoting $V(\widetilde{\mathcal{Q}}_{\Gamma \cup \{A\}})$, $V(\mathcal{B}_{\Gamma \cup \{A\}})$ and $V(\mathcal{D}_A)$ to be the sets of propositional variables occurring in the classes:

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Corollary

Suppose $\Gamma \vdash_c A$ and $V(\widetilde{\mathcal{Q}}_{\Gamma\cup\{A\}}) \subseteq V(\mathcal{D}_A)$ or $V(\mathcal{B}_{\Gamma\cup\{A\}}) \subseteq V(\mathcal{D}_A)$. Then $\widetilde{\mathcal{R}}_A, \mathcal{D}_A, \Gamma \vdash_m A$.

Future directions

- Is it possible to use classes of principles weaker than WLEM and DNE?
- Can we extend Ishihara's class for Glivenko's logic and beyond?

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