Effective Wadge hierarchy in computable quasi-Polish spaces

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Mathematical Logic and its Applications Workshop

March 22, 2021

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Introduction

The classical Borel, Luzin, and Hausdorff hierarchies in Polish spaces, which are defined using set operations, play an important role in descriptive set theory (DST). In 2013 these hierarchies were extended and shown to have similar nice properties also in quasi-Polish spaces which include many non-Hausdorff spaces of interest for several branches of mathematics and theoretical computer science (M. de Brecht).

The Wadge hierarchy is non-classical in the sense that it is based on a notion of reducibility that was not recognized in the classical DST, and on using ingenious versions of Gale-Stewart games rather than on set operations. For subsets A, B of the Baire space $\mathcal{N} = \omega^\omega$, A is *Wadge reducible* to B ($A \leq_W B$), if $A = f^{-1}(B)$ for some continuous function f on $\mathcal{N}$. The quotient-poset of the preorder $(P(\mathcal{N}); \leq_W)$ under the induced equivalence relation $\equiv_W$ on the power-set of $\mathcal{N}$ is called *the structure of Wadge degrees* in $\mathcal{N}$.

Introduction

W. Wadge characterised the structure of Wadge degrees of Borel sets (i.e., the quotient-poset of $(\mathsf{B}(\mathcal{N}); \leq_W)$) up to isomorphism. In particular, this quotient-poset is semi-well-ordered, hence it is well-founded and has no 3 pairwise incomparable elements. Under some set-theoretic assumptions, R. van Wesep extended the WH to arbitrary subsets of Baire space. Here we will stay in the realm of Borel sets.

The Wadge hierarchy $\{\Sigma_\alpha(\mathcal{N})\}_{\alpha<\upsilon}$ (for a rather large ordinal υ) in $\mathcal{N}$ is a great refinement of the Borel hierarchy. It was originally defined only for the Baire space but the structure $(\mathsf{B}(X); \leq_W)$ of Wadge degrees of Borel sets in any zero-dimensional Polish space X remains semi-well-ordered.

Introduction

In this work we attempt to find the "correct" extension of the Wadge hierarchy from Polish zero-dimensional spaces to arbitrary second countable spaces, with the emphasis to quasi-Polish spaces. We prove properties of the resulting hierarchy which demonstrate correctness. We also develop an effective version of this theory for the effective cb_0-spaces, with the emphasis to computable quasi-Polish spaces.

There are at least three approaches to the problem of extension the WH. The first approach is to show that Wadge reducibility in such spaces behaves similarly to its behaviour in the Baire space, i.e. it is a semi-well-order. Unfortunately, this is not the case: for many natural quasi-Polish spaces X the structure $(\mathrm{B}(X); \leq_W)$ is not well-founded and has antichains with more than 2 elements. Thus, this approach does not lead to a reasonable extension of the Wadge hierarchy to quasi-Polish spaces.

Introduction

The second approach, independently suggested by Y. Pequignot and myself, is based on the characterization of quasi-Polish spaces as the second countable T_0-spaces X such that there is a total admissible representation ξ from $\mathcal{N}$ onto X. Namely, one can *define* the Wadge hierarchy $\{\Sigma_\alpha(X)\}_{\alpha<\upsilon}$ in X by $\Sigma_\alpha(X)=\{A\subseteq X\mid \xi^{-1}(A)\in\Sigma_\alpha(\mathcal{N})\}$. This definition is short and elegant but it gives no real understanding of how the levels $\Sigma_\alpha(X)$ look like.

The third approach consists in set-theoretic description of subsequent refinements of the Borel hierarchy. It was thoroughly studied by M. de Brecht for the Borel and Hausdorff hierarchies in quasi-Polish spaces and extended by me to some other levels of the WH.

Introduction

Here we propose a set-theoretic definition for the whole Wadge hierarchy of Borel sets from the second approach. The definition is an infinitary version of the so called fine hierarchy introduced and studied in a series of my publications. In fact, we develop a "classical" infinitary version of the FH. Arguably, our infinitary fine hierarchy (IFH), and hence also the Wadge hierarchy, is a kind of "iterated difference hierarchy" over levels of the Borel hierarchy; it only remains to make precise how to "iterate" the difference hierarchies.

We also develop an effective finitary version of the Wadge hierarchy in effective spaces and computable quasi-Polish spaces. This version is just a particular case of the FH.

Introduction

Along with describing (hopefully) the right version of the Wadge hierarchy (by identifying it with the IFH) in arbitrary spaces we show that it behaves well in second countable spaces and especially in quasi-Polish spaces. E.g., it provides the description of all levels $\Sigma_\alpha(X)$ in quasi-Polish spaces. Also, all levels of the IFH are preserved by continuous open surjections between second countable spaces

This gives a broad extension of results by Saint Raymond and de Brecht for the Borel and Hausdorff hierarchies, and new results on the non-collapse of the WH in different spaces. Similar results are obtained for the EWH.

Introduction

Notions and results of this paper apply not only to the Wadge hierarchy of sets discussed so far but also to a more general hierarchy of functions $A : X \to Q$ from a space X to an arbitrary quasiorder Q. We identify such functions with *Q-partitions of X* of the form $\{A^{-1}(q)\}_{q \in Q}$ in order to stress their close relation to k-partitions (obtained when $Q = \bar{k} = \{0, \ldots, k-1\}$ is an antichain with k-elements) studied by many authors.

For Q-partitions A, B of X, let $A \leq_W B$ mean that there is a continuous function f on X such that $A(x) \leq_Q B(f(x))$ for each $x \in X$. The case of sets corresponds to the case of 2-partitions.

Introduction

Let $\mathrm{B}(Q^X)$ be the set of Borel Q-partitions A (for which $A^{-1}(q) \in \mathrm{B}(X)$ for all $q \in Q$). A celebrated theorem of van Engelen, Miller and Steel shows that if Q is a countable better quasiorder (bqo) then $\mathcal{W}_Q = (\mathrm{B}(Q^{\mathcal{N}}); \leq_W)$ is a bqo. Although this theorem gives an important information about the quotient-poset of $\mathcal{W}_Q$, it is far from a characterisation.

Recently, T. Kihara and A. Montalbán gave a complete characterization of the latter structure which opened new possibilities for the Wadge theory. Our definitions and many of our results hold in this more general context.

Effective spaces

All considered spaces are assumed to be countably based T_0 (sometimes we call such spaces cb$_0$-spaces). By *effectivization of a cb$_0$-space* X we mean a numbering $\beta : \omega \to P(X)$ of a base in X such that there is a uniform sequence $\{A_{ij}\}$ of c.e. sets with $\beta_i \cap \beta_j = \bigcup \beta(A_{ij})$, where $\beta(A)$ is the image of A under β. The numbering β is called an *effective base of* X while the pair (X, β) is called an effective space. We simplify (X, β) to X if β is clear from the context. *Effectively open sets* in X are the sets of the form $\bigcup_{i \in W} \beta(i) = \bigcup \beta(W)$, for some c.e. set $W \subseteq \mathbb{N}$. The standard numbering $\{W_n\}$ of c.e. sets induces a numbering of the effectively open sets.

A function $f : (X, \beta) \to (Y, \gamma)$ is *computable* if $f^{-1}(\gamma_n) = \bigcup \beta(W_{g(n)})$ for some computable function $g : \mathbb{N} \to \mathbb{N}$. A function $f : (X, \beta) \to (Y, \gamma)$ is *effectively open* if $f(\beta_n) = \bigcup \gamma(W_{h(n)})$ for some computable function $h : \mathbb{N} \to \mathbb{N}$.

Computable quasi-Polish spaces

Let $\{\Sigma^0_{1+n}(X)\}_{n<\omega}$ be the effective Borel hierarchy in an effective space. As usual, levels of the effective hierarchies are denoted in the same manner as levels of the corresponding classical hierarchies, using the lightface letters Σ, Π, Δ instead of the boldface $\boldsymbol{\Sigma}, \boldsymbol{\Pi}, \boldsymbol{\Delta}$ used for the classical hierarchies. Though the effective hierarchies are naturally defined in arbitrary effective space, some important properties only hold for special classes of spaces. Recall a similar situation in classical DST where the spaces with "good" DST (namely, the quasi-Polish spaces) were identified relatively recently.

Effectivizing one of several characterization of QP-spaces we obtain the following notion. By a *computable quasi-Polish space* we mean an effective space (X, β) such that there exists a computable effectively open surjection $\xi : \mathcal{N} \to X$ from the Baire space onto (X, β).

Labeled trees

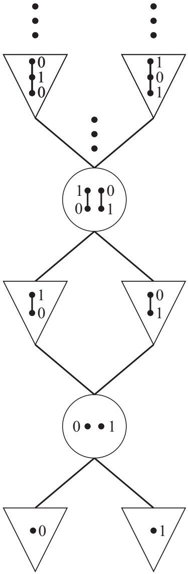
Let $(Q;\leq)$ be a preorder. A *Q-tree* is a pair (T,t) consisting of a finite tree $T\subseteq\omega^*$ and a labeling $t:T\to Q$. Let $\mathcal{T}_Q$ denote the set of all finite Q-trees. The *h-preorder* $\leq_h$ on $\mathcal{T}_Q$ is defined as follows: $(T,t)\leq_h(S,s)$, if there is a monotone function $f:(T;\sqsubseteq)\to(S;\sqsubseteq)$ satisfying $\forall x\in T(t(x)\leq s(f(x)))$. Though many results of this paper may be extended to arbitrary finite preorders Q in place of $\bar{k}$, we will mainly stick to k-partitions in order to avoid some complications and exceptions.

The preorder Q is a *well quasiorder* (WQO) if it has neither infinite descending chains nor infinite antichains. An example of WQO is the antichain $\bar{k}$ with k elements. A famous Kruskall's theorem implies that if Q is WQO then $(\mathcal{T}_Q;\leq_h)$ is WQO.

Iterated labeled trees

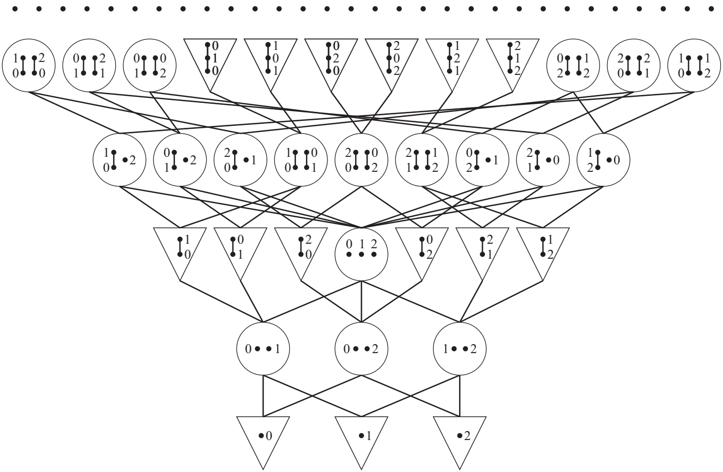
Define the sequence $\{\mathcal{T}_k(n)\}_{n<\omega}$ of preorders by induction on n as follows: $\mathcal{T}_k(0)=\overline{k}$ and $\mathcal{T}_k(n+1)=\mathcal{T}_{\mathcal{T}_k(n)}$. The sets $\mathcal{T}_k(n)$, $n<\omega$, are pairwise disjoint but, identifying the elements i of $\overline{k}$ with the corresponding singleton trees $s(i)$ labeled by i (which are precisely the minimal elements of $\mathcal{T}_k(1)$), we may think that $\mathcal{T}_k(0)\sqsubseteq\mathcal{T}_k(1)$, i.e. the quotient-poset of the first preorder is an initial segment of the quotient-poset of the other. This also induces an embedding of $\mathcal{T}_k(n)$ into $\mathcal{T}_k(n+1)$ as an initial segment, so (abusing notation) we may think that $\mathcal{T}_k(0)\sqsubseteq\mathcal{T}_k(1)\sqsubseteq\cdots$, hence $\mathcal{T}_k(\omega)=\bigcup_{n<\omega}\mathcal{T}_k(n)$ is WQO w.r.t. the induced preorder which we also denote $\leq_h$. The embedding s is extended to $\mathcal{T}_k(\omega)$ by defining $s(T)$ as the singleton tree labeled by T.

Iterated labeled trees



An initial segment of $(\mathcal{F}_2(1); \leq_h)$.

Iterated labeled trees



An initial segment of $(\mathcal{F}_3(1); \leq_h)$.

The fine hierarchy

By a *base in a set* X we mean a sequence $\mathcal{L} = \{\mathcal{L}_n\}_{n<\omega}$ of subclasses of $P(X)$ such that any $\mathcal{L}_n$ is closed under finite union and intersection, contains $\emptyset, X$ and satisfies $\mathcal{L}_n \cup \check{\mathcal{L}}_n \subseteq \mathcal{L}_{n+1}$. For this paper, the *effective Borel bases* $\mathcal{L}(X) = \{\Sigma^0_{1+n}(X)\}$ in effective spaces X are especially relevant.

With any base $\mathcal{L}(X)$ we associate some other bases as follows. For any $m < \omega$, let $\mathcal{L}^m(X) = \{\mathcal{L}_{m+n}(X)\}_n$; we call this base the *m-shift of* $\mathcal{L}(X)$. For any $U \in \mathcal{L}_0$, let $\mathcal{L}(U) = \{\mathcal{L}_n(U)\}_{n<\omega}$ where $\mathcal{L}_n(U) = \{U \cap S \mid S \in \mathcal{L}_n(X)\}$; we call this base the *U-restriction of* $\mathcal{L}(X)$.

With any base $\mathcal{L} = \{\mathcal{L}_n\}_{n<\omega}$ in X we associate the *fine hierarchy of k-partitions over* $\mathcal{L}$ which is a family $\{\mathcal{L}(X, T)\}_{T \in \mathcal{T}_k(\omega)}$ of subsets of k^X. Since $T \leq_h S$ implies $\mathcal{L}(X, T) \subseteq \mathcal{L}(X, S)$, $(\{\mathcal{L}(X, T) \mid T \in \mathcal{T}_k(\omega)\}; \subseteq)$ is a WQO.

The fine hierarchy

The infinitary FH looks similarly, only the base now is the whole Borel hierarchy, and instead of finite trees we take infinite well-founded trees which leads to much longer iterations of the construction $Q \mapsto \mathcal{T}_Q$.

The FH of sets is obtained from this construction for $k = 2$ since the quotient-poset of $(\mathcal{T}_2(\omega); \leq_h)$ has order type $\bar{2} \cdot \varepsilon_0$.

For any finite tree $T \subseteq \omega^*$ we consider T-families $\{U_\tau\}_{\tau \in T}$ of $\mathcal{L}_n$-sets such that $U_\varepsilon = X$ and $U_\tau \supseteq U_{\tau'}$ for all $\tau \sqsubseteq \tau' \in T$. The T-family $\{\tilde{U}_\tau\}$, where $\tilde{U}_\tau = U_\tau \setminus \bigcup\{U_{\tau'} \mid \tau \sqsubset \tau' \in T\}$, is the corresponding tree of components. Such families may determine a k-partition A of X by the usual mind-change procedure (given $x \in X$, we think $A(x) = t(\varepsilon)$ while $x \in \tilde{U}_\varepsilon$; if x moves down to some U_i, we start to think that $A(x) = t(i)$ and so on). Note that this time there is a danger of inconsistency if $x \in U_i \cap U_j$ for some distinct $i, j \in T$; we just define $A(x)$ only if there are no inconsistencies in the lowest components containing x.

Iterating this mind-change idea, we define, for $T \in \mathcal{T}_k(\omega)$, the notion "$F$ is a T-family in $\mathcal{L}(X)$" by induction:
1) If $T \in \mathcal{T}_k(0)$ then $F = \{X\}$.
2) If $(T, t) \in \mathcal{T}_k(n+1)$ then $F = (\{U_\tau\}, \{F_\tau\})$ where $\{U_\tau\}$ is a monotone T-family of $\mathcal{L}_0$-sets with $U_\varepsilon = X$ and, for each $\tau \in T$, F_τ is a $t(\tau)$-family in $\mathcal{L}^1(\tilde{U}_\tau)$.
We also define the notion "a T-family F in $\mathcal{L}(X)$ determines a partition $A : X \to \bar{k}$" by induction:
1) If $T \in \mathcal{T}_k(0)$, $T = i < k$ (so $F = \{X\}$), then T determines the constant partition $A = \lambda x.i$.
2) If $(T, t) \in \mathcal{T}_k(n+1)$ (so F is of the form $(\{U_\tau\}, \{F_\tau\})$) then T determines the k-partition A such that $A|_{\tilde{U}_\tau} = B_\tau$ for every $\tau \in T$, where $B_\tau : \tilde{U}_\tau \to \bar{k}$ is the k-partition of $\tilde{U}_\tau$ determined by F_τ.

Finally, let $\mathcal{L}(X, T)$ be the set of $A : X \to \bar{k}$ determined by some T-family in $\mathcal{L}(X)$.

The fine hierarchy

Let us give examples of explicit descriptions of the introduced notions. For $T = i \in \mathcal{T}_k(0)$, there is only one T-family $\{X\}$ in $\mathcal{L}(X)$ which determines the constant partition $\lambda x.i$. For $T \in \mathcal{T}_k(1)$, a T-family F in $\mathcal{L}(X)$ is essentially a family $\{U_\tau\}$ of $\mathcal{L}_0(X)$-sets whose components $\tilde{U}_\tau$ cover X. Such a family determines A if $A(x) = t(\tau)$, for any $\tau \in T$ with $x \in \tilde{U}_\tau$. Note that $t : T \to \bar{k}$ and that x may belong to different components $\tilde{U}_\tau$, $\tilde{U}_\sigma$ with incomparable τ, σ.

For $T \in \mathcal{T}_k(2)$, a T-family F in $\mathcal{L}(X)$ consists of a family $\{U_\tau\}$ as above, and, for each $\tau_0 \in T$, a family $\{U_{\tau_0\tau_1}\}_{\tau_1 \in t_0(\tau_0)}$ of $\mathcal{L}_1(X)$-sets whose components (which we call second-level components) $\tilde{U}_{\tau_0\tau_1}$ cover $\tilde{U}_{\tau_0}$ (called first-level components). Such an F determines A if $A(x) = t_1(\tau_1)$, for all $\tau_0 \in T, \tau_1 \in t_0(\tau_0)$ with $x \in \tilde{U}_{\tau_0\tau_1}$. Note that $t_0 : T \to \mathcal{T}_k(1), t_1 : t_0(\tau_0) \to \bar{k}$.

The fine hierarchy

For $T \in \mathcal{T}_k(3)$, a T-family F in $\mathcal{L}(X)$ consists of families $\{U_\tau\}, \{U_{\tau_0\tau_1}\}$ as above and, for all $\tau_0 \in T, \tau_1 \in t_0(\tau_0)$, a family $\{U_{\tau_0\tau_1\tau_2}\}_{\tau_2 \in t_1(\tau_1)}$ of $\mathcal{L}_2(X)$-sets whose components $\tilde{U}_{\tau_0\tau_1\tau_2}$ of the third level cover $\tilde{U}_{\tau_0\tau_1}$. Such F determines A if $A(x) = t_2(\tau_2)$, for all $\tau_0 \in T, \tau_1 \in t_0(\tau_0), \tau_2 \in t_1(\tau_1)$ with $x \in \tilde{U}_{\tau_0\tau_1\tau_2}$. Note that $t_0 : T \to \mathcal{T}_k(2), t_1 : t_0(\tau_0) \to \mathcal{T}_k(1), t_2 : t_1(\tau_1) \to \overline{k}$.

Thus, the T-family F in an effective Borel base that determines A, provides a mind-change algorithm for computing $A(x)$ for a given $x \in X$ as follows. First, we search for a component $\tilde{U}_{\tau_0}$ containing x; this is the usual mind-change procedure working with differences of Σ^0_1-sets. While x sits in $\tilde{U}_{\tau_0}$, we search for a component $\tilde{U}_{\tau_0\tau_1}$ containing x; this is a harder mind-change procedure working with differences of Σ^0_2-sets, and so on.

Preservation property

A principal result is the following preservation property for levels of the EWH.

T h e o r e m. Let $f : X \to Y$ be a computable effectively open surjection between effective spaces and $A : Y \to \bar{k}$. Then for any $T \in \mathcal{T}_k(\omega)$ we have: $A \in \Sigma(Y, T)$ iff $A \circ f \in \Sigma(X, T)$. In particular, for all $A \subseteq Y$ and $\alpha < \varepsilon_0$ we have: $A \in \Sigma_\alpha(Y)$ iff $f^{-1}(A) \in \Sigma_\alpha(X)$. The non-effective version of this theorem for the IFH also holds.

This theorem extends the earlier results of Saint-Raymond and de Brecht for the Borel hierarchy, and of Callard-Hoyrup for the effective Borel hierarchy, utilizing the Baire property for cb_0-spaces and its effective version for effective spaces.

It applies to (computable) quasi-Polish spaces which are precisely the (computable versions of) continuous open images of the Baire space. It provides a general tool to deal with some properties of the (effective) WH (and thus for many other hierarchies).

Hausdorff-Kuratowski-type theorems

We say that an effective space X *satisfies n-HK theorem* if $\Delta^0_{n+2}(k^X)=\bigcup\{\Sigma(X,s^n(T))\mid T\in\mathcal{T}^*_k(1)\}$ where $\Delta^0_{n+2}(k^X)$ is the set of $A\in k^X$ with components $A_0,\ldots,A_{k-1}$ in $\Delta^0_{n+2}(X)$, and s^n is the nth iteration of the function s forming the singleton trees. For $n=0$ the equality simplifies to $\Delta^0_2(k^X)=\bigcup\{\Sigma(X,T)\mid T\in\mathcal{T}^*_k(1)\}$ which we call *the effective Hausdorff theorem for k-partitions*.

T h e o r e m. If an effective space X satisfies n-HK theorem and $Y\leq_{ceo}X$ then so does Y. Thus, if $\mathcal{N}$ satisfies n-HK theorem then so does every CQP-space.

C o r o l l a r y. Every CQP-space satisfies the effective Hausdorff theorem for k-partitions.

Many more corollaries hold in the non-effective case because the structure of WH implies a lot of HK-type theorems in the Baire space which are inherited by every quasi-Polish space, according to the preservation property.

Non-collapse property

Hierarchies are basic tools for calibrating objects according to their complexity, hence the non-collapse of a natural hierarchy is fundamental for understanding the corresponding notion of complexity. The preservation property provides a tool for proving the non-collapse of the (effective) WH (hence also for other hierarchies).

We say that EWH $\{\Sigma(X,T)\}_{T\in\mathcal{T}_\omega(\bar{k})}$ *does not collapse at level* T if $\Sigma(X,T)\not\subseteq\Sigma(X,V)$ for each $V\in\mathcal{T}_\omega(\bar{k})$ with $T\not\leq_h V$; it *strongly does not collapse at level* T if $\Sigma(X,T)\not\subseteq\bigcup\{\Sigma(X,V)\mid V\in\mathcal{T}_\omega(\bar{k}),\ T\not\leq_h V\}$. We say that $\{\Sigma(X,T)\}_{T\in\mathcal{T}_\omega(\bar{k})}$ *(strongly) does not collapse* if it (strongly) does not collapse at any level $T\in\mathcal{T}_\omega(\bar{k})$. The latter non-strong version is equivalent to saying that the quotient-poset of $(\mathcal{T}_\omega(\bar{k});\leq_h)$ is isomorphic to $(\{\Sigma(X,T)\mid T\in\mathcal{T}_\omega(\bar{k})\};\subseteq)$.

Non-collapse property

The non-collapse for the boldface versions are defined in the same way. In the effective case, there are also the following uniform versions of non-collapse property which relate EWH to the corresponding WH. The EWH $\{\Sigma(X,T)\}$ *uniformly does not collapse at level* T if $\Sigma(X,T) \not\subseteq \Sigma(X,V)$ for each $V \in \mathcal{T}_\omega(\bar{k})$ with $T \not\leq_h V$. It *strong uniformly does not collapse* at level T if $\Sigma(X,T) \not\subseteq \bigcup\{\Sigma(X,V) \mid V \in \mathcal{T}_\omega(\bar{k}),\ T \not\leq_h V\}$. It *strong uniformly does not collapse* if
$\Sigma(X,T) \not\subseteq \bigcup\{\Sigma(X,V) \mid V \in \mathcal{T}_\omega(\bar{k}),\ T \not\leq_h V\}$ for all $T \in \mathcal{T}_\omega(\bar{k})$.

For the case of sets $k=2$ these definitions are equivalent to the standard definition of non-collapse in DST (Σ-levels are distinct from the corresponding Π-levels), and the strong version is equivalent to the non-strong one.

Non-collapse property

For cb_0-spaces X and Y, let $X \leq_{co} Y$ mean that there is a continuous open surjection f from Y onto X. For effective cb_0-spaces X and Y, let $X \leq_{eco} Y$ mean that there is a computable effectively open surjection f from Y onto X. Clearly, both $\leq_{eco}$ and $\leq_{co}$ are preorders, and the first preorder is contained in the second.

The HK-type theorems are inherited downwards w.r.t. the introduced preorders. The non-collapse property is inherited upwards:

Non-collapse property

P r o p o s i t i o n. 1. If $X \leq_{co} Y$ and $\{\Sigma(X,T)\}_{T\in\mathcal{T}_\omega(\bar{k})}$ (strongly) does not collapse (at level T) then $\{\Sigma(Y,T)\}$ (strongly) does not collapse (at level T). The same holds for the infinitary version of WH in X.

2. If $X \leq_{eco} Y$ and $\{\Sigma(X,T)\}_{T\in\mathcal{T}_\omega(\bar{k})}$ (strongly) does not collapse (at level T) then $\{\Sigma(Y,T)\}$ (strongly) does not collapse (at level T). The same holds for the uniform version of non-collapse property.

Non-collapse property

C o r o l l a r y. 1. If X is quasi-Polish and $\{\Sigma(X,T)\}_{T\in\mathcal{T}_\omega(\bar{k})}$ (strongly) does not collapse (at level T) then $\{\Sigma(\mathcal{N},T)\}$ (strongly) does not collapse (at level T). The same for IWH.
2. If X is computable quasi-Polish and $\{\Sigma(X,T)\}_{T\in\mathcal{T}_\omega(\bar{k})}$ (strongly) does not collapse (at level T) then $\{\Sigma(\mathcal{N},T)\}$ (strongly) does not collapse (at level T). The same holds for the uniform version.
3. If X is the product of a sequence $\{X_n\}$ of nonempty cb$_0$-spaces, and the finitary WH $\{\Sigma(X_n,T)\}_{T\in\mathcal{T}_\omega(\bar{k})}$ (strongly) does not collapse (at level T) for some $n<\omega$, then $\{\Sigma(X,T)\}$ (strongly) does not collapse (at level T). The same for IWH.
4. If X is the product of a uniform sequence $\{X_n\}$ of nonempty effective cb$_0$-spaces, and $\{\Sigma(X_n,T)\}_{T\in\mathcal{T}_\omega(\bar{k})}$ (strongly) does not collapse (at level T) for some $n<\omega$, then $\{\Sigma(X,T)\}$ (strongly) does not collapse (at level T). The same holds for the uniform version.

Non-collapse property

Although the assertion (1) is void, it is of methodological interest because it shows that proving the non-collapse of WH in any quasi-Polish space is at least as complicated as proving it in $\mathcal{N}$, and the proof of the latter fact is highly non-trivial. The same applies to item (2) but this assertion is non-void because the non-collapse of EWH in $\mathcal{N}$ was open until this work, to my knowledge.

T h e o r e m. 1. The WH $\{\Sigma(\omega^{\leq\omega}, T)\}_{T\in\mathcal{T}_\omega(\bar{k})}$ strongly does not collapse. Similarly for the infinitary WH.
2. The EWH $\{\Sigma(\omega^{\leq\omega}, T)\}_{T\in\mathcal{T}_\omega(\bar{k})}$ strong uniformly does not collapse.
3. The EWHs $\{\Sigma(\mathcal{N}, T)\}$ and $\{\Sigma(\mathcal{C}, T)\}_{T\in\mathcal{T}_\omega(\bar{k})}$ strong uniformly do not collapse.
4. The EWH $\{\Sigma(\mathbb{N}, T)\}_{T\in\mathcal{T}_\omega(\bar{k})}$ strongly does not collapse.
5. The EWH $\{\Sigma(\mathbb{N}_\perp, T)\}_{T\in\mathcal{T}_\omega(\bar{k})}$ strongly does not collapse.
6. The EWH $\{\Sigma(\mathcal{C}, T)\}_{T\in\mathcal{T}_\omega(\bar{k})}$ strongly does not collapse.

Conclusion

The non-collapse of EWH is highly non-trivial already for the discrete space $\mathbb{N}$ of natural numbers. In fact, for the case of sets it follows from my results of 1983. For k-partitions with $k > 2$, the non-collapse property was not proved in that paper because that time we did not have a convincing notion of a hierarchy of k-partitions. The non-collapse of EWH in $\mathcal{N}$ provides an effective version for the fundamental result of Kihara and Montalbán; modulo their proof, our proof is easy.

The preservation property suggests a method for proving non-collapse in different spaces. Unfortunately, this method is less general than the dual inheritance method for proving the Hausdorff-Kuratowski-type theorems that completely reduces this property in (computable) quasi-Polish spaces to that in the Baire space. Nevertheless, the method of proving non-collapse provides some insight which enables e.g. to show that the non-collapse property is hard to prove for the majority of spaces.

THANK YOU FOR YOUR ATTENTION!!

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