The Compact Hyperspace Monad, a Constructive Approach[1]

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1. Monads

Definition

Let $\mathcal{C}$ be a category. A monad consists of an endofunctor $\mathcal{F}\colon \mathcal{C} \to \mathcal{C}$ together with two natural transformations $\eta\colon \mathrm{Id}_{\mathcal{C}} \to \mathcal{F}$ and $\mu\colon \mathcal{F}^2 \to \mathcal{F}$ so that for any $X \in \mathcal{C}$,

$$\mu_X \circ \mathcal{F}(\mu_X) = \mu_X \circ \mu_{\mathcal{F}(X)} \quad \text{and} \quad \mu_X \circ \eta_{\mathcal{F}(X)} = \mu_X \circ \mathcal{F}(\eta_X) = \mathrm{Id}_{\mathcal{F}(X)}\,.$$

Example

Let $\mathcal{P}\colon \mathbf{SET} \to \mathbf{SET}$ be the powerset functor. Define

$$\eta_X(x) := \{x\} \quad \text{and} \quad \mu_X(\mathbb{X}) = \bigcup \mathbb{X},$$

where X is a set and $\mathbb{X} \in \mathcal{P}(\mathcal{P}(X))$. Then $(\mathcal{P}, \eta, \mu)$ is a monad.

Theorem (E. Michael (1951))

Let (X, τ) be compact Hausdorff and $\mathcal{K}(X)$ the set of all non-empty compact subsets of X. Then:

1. *$\mathcal{K}(X)$ is compact Hausdorff w.r.t. the Vietoris topology τ_V, generated by the sets*

$$[U; V_1, \ldots, V_n] := \{ A \in \mathcal{K}(X) \mid A \subseteq U \wedge (\forall 1 \leqslant i \leqslant n) A \cap V_i \neq \varnothing \},$$

   for $U, V_1, \ldots, V_n$ open in X.

2. *For compact subsets $\mathbb{K}$ of $\mathcal{K}(X)$, $\bigcup \mathbb{K}$ is compact.*

Lemma

For compact metric space X, $\mathcal{K}(X)$ is a compact metric space with the Hausdorff metric. The metric topology coincides with the Vietoris topology.

Let **CM** be the category of all compact metric spaces with continuous maps as morphisms. For morphisms $f\colon X \to Y$ let

$$\mathcal{K}(f)(A) := f[A],$$

that is, $\mathcal{K}(f)(A)$ the direct image of A under f. Then it follows for η and μ as above that $(\mathcal{K}, \eta, \mu)$ is a monad.

Our aim is to derive an analogous result in a framework that allows to compute with the elements of the spaces under consideration. We will follow the line of research of U. Berger et al. in which one works in an intuitionistic logic extended by inductive and co-inductive definitions.

Spaces under consideration will have a co-inductive characterisation from which by a suitable realisability interpretation trees representing the elements of the spaces can be extracted with which one computes.

2. Iterated function systems

Let (X, δ) be a compact metric space and D be a finite set of contractions $d \colon X \to X$. Then (X, D) is called iterated function system (IFS).

Definition

An IFS (X, D) is said to be covering if

$$X = \bigcup \{\, \mathrm{range}(d) \mid d \in D \,\}.$$

The covering condition allows to characterise X co-inductively.

Definition

Define $\mathbb{C}_X$ co-inductively to be the largest subset of X such that for all $x \in X$,

$$x \in \mathbb{C}_X \to (\exists d \in D)(\exists y \in \mathbb{C}_X)\, x = d(y).$$

Lemma

Let (X, D) *be a covering IFS. Then*

$$X = \mathbb{C}_X.$$

Proof.

By definition, $\mathbb{C}_X \subseteq X$. The converse inclusion follows by co-induction. Because (X, D) is covering, the defining implication of $\mathbb{C}_X$ remains correct if $\mathbb{C}_X$ is replaced by X. □

The existential quantifiers in the definition of $\mathbb{C}_X$ need to be interpreted constructively. Then by applying the definition of $\mathbb{C}_X$ again and again, one obtains a sequence $d_0, d_1, \ldots$ of maps in D so that

$$x \in \bigcap_n \mathrm{range}(d_0 \circ \cdots \circ d_{n-1}).$$

Note that since X is compact,

$$\bigcap_n \mathrm{range}(d_0 \circ \cdots \circ d_{n-1}) \neq \varnothing.$$

Contractivity of the d, on the other hand, implies that

$$\|\bigcap_n \mathrm{range}(\alpha_0 \circ \cdots \circ \alpha_{n-1})\| \leqslant 1.$$

Hence, the sequence $\alpha := d_0, d_1, \ldots$ uniquely determines x.

3. The case $\mathcal{K}(X)$

In case of points x, we characterised x by determining a $d \in D$ with $x \in \mathrm{range}(d)$. In the case of compact sets K, we will characterise K by a finite subset D' of D such that $d \in D'$ exactly if K hits $\mathrm{range}(d)$.

For $d_1, \ldots, d_r \in D$ define

$$[d_1, \ldots, d_r](K_1, \ldots, K_r) := \bigcup_{i=1}^{r} d_i[K_i] = \bigcup_{i=1}^{r} \mathcal{K}(d_i)(K_i).$$

Note that $[d_1, \ldots, d_r] \colon \mathcal{K}(X)^r \to \mathcal{K}(X)$ is contracting. Let

$$\mathcal{K}(D) := \{ [d_1, \ldots, d_r] \mid d_1, \ldots, d_r \in D \text{ pairwise distinct} \}.$$

Then $\mathcal{K}(D)$ is finite and $(\mathcal{K}(X), \mathcal{K}(D))$ is an extended IFS.

Lemma

If (X, D) *is covering, so is* $(\mathcal{K}(X), \mathcal{K}(D))$.

Definition

Define $\mathbb{C}_{\mathcal{K}(X)}$ co-inductively to be the largest subset of $\mathcal{K}(X)$ such that for all $K \in \mathcal{K}(X)$,

$$K \in \mathbb{C}_{\mathcal{K}(X)} \to (\exists [d_1, \ldots, d_r] \in \mathcal{K}(D)) \\ (\exists K_1, \ldots, K_r \in \mathbb{C}_{\mathcal{K}(X)})\, K = [d_1, \ldots, d_r](K_1, \ldots, K_r).$$

Lemma

Let (X, D) *be a covering IFS. Then*

$$\mathcal{K}(X) = \mathbb{C}_{\mathcal{K}(X)}.$$

4. Products

Let $X_1, \dots, X_n$ be compact metric spaces and $X_1 \times \cdots \times X_n$ endowed with the maximum metric. Then $X_1 \times \cdots \times X_n$ is compact as well.

Assume that (X_1, D_1), ..., (X_n, D_n) are extended IFS and let

$$s_D := \max \{ \mathrm{ar}(d) \mid d \in \bigcup_{i=1}^n D_i \}.$$

Replace $d \in D_i$ by $\hat{d}$ with

$$\hat{d}(x_1, \dots, x_{s_D}) := d(x_1, \dots, x_{\mathrm{ar}(d)}),$$

for $x_1, \dots, x_{s_D} \in X_i$. For $d_i \in D_i$ $(1 \leqslant i \leqslant n)$ define

$$\begin{aligned}\langle d_1, \dots, d_n \rangle ((x_1^{(1)}, \dots, x_n^{(1)}), \dots, (x_1^{(s_D)}, \dots, x_n^{(s_D)})) \\ := (d_1(x_1^{(1)}, \dots, x_1^{(s_D)}), \dots, d_n(x_n^{(1)}, \dots, x_n^{(s_D)})).\end{aligned}$$

Set

$$\Pi_{i=1}^{n} D_i := \{ \langle d_1, \ldots, d_n \rangle \mid (d_1, \ldots, d_n) \in \bigtimes_{i=1}^{n} D_i \}.$$

Proposition

Let $(X_1, D_1), \ldots, (X_n, D_n)$ be extended IFS. Then also

$$\bigtimes_{i=1}^{n} (X_i, D_i) := (\bigtimes_{i=1}^{n} X_i, \Pi_{i=1}^{n} D_i)$$

is an extended IFS. Moreover,

1. *If (X_i, D_i) is compact, for all $1 \leqslant i \leqslant n$, so is $\bigtimes_{i=1}^{n}(X_i, D_i)$.*
2. *If (X_i, D_i) is covering, for all $1 \leqslant i \leqslant n$, so is $\bigtimes_{i=1}^{n}(X_i, D_i)$.*

5. The case $\mathcal{K}(\mathcal{K}(X))$

In order to obtain a finite set of covering maps it seems natural to iterate the above construction. Consider

$$[[d_1^{(1)}, \dots, d_{r_1}^{(1)}], \dots, [d_1^{(n)}, \dots, d_{r_n}^{(n)}]](\mathbb{K}_1, \dots, \mathbb{K}_n) := \bigcup_{i=1}^{n} \mathcal{K}([d_1^{(i)}, \dots, d_{r_i}^{(i)}])(\mathbb{K}_i).$$

A set $\mathbb{K} \in \mathcal{K}^2(X)$ is covered by the range of this map, just if its elements K are such that for some $1 \leqslant i \leqslant n$, K hits exactly the ranges of the maps $d_1^{(i)}, \dots, d_{r_i}^{(i)}$.

The map $[[d_1^{(1)}, \dots, d_{r_1}^{(1)}], \dots, [d_1^{(n)}, \dots, d_{r_n}^{(n)}]]$ has type

$$\mathcal{K}(\mathcal{K}(X)^{r_1}) \times \cdots \times \mathcal{K}(\mathcal{K}(X)^{r_n}) \to \mathcal{K}^2(X).$$

Thus $(\mathcal{K}^2(X), \mathcal{K}^2(D))$ is not an extended IFS.

Assume that (X, D) is a compact IFS with $\mathrm{ar}(d) = 1$ $(d \in D)$. Then the maximal arity of some $\vec{d} \in \mathcal{K}(D)$ is $m(D) := \|D\|$.

We write

$$\mathcal{K}(X) \xleftarrow[\mathcal{K}(D)]{} \mathcal{K}(X)^{m(D)}$$

to mean that all $\vec{d} \in \mathcal{K}(D)$ are of type $\mathcal{K}(X)^{m(D)} \to \mathcal{K}(X)$.

By definition of the product we have that

$$\mathcal{K}(X)^{m(D)} \xleftarrow[\Pi(\mathcal{K}(D)^{m(D)})]{} (\mathcal{K}(X)^{m(D)})^{m(D)}.$$

Proceeding in this way we obtain a co-chain

$$\mathcal{K}(X)_0 \xleftarrow[\mathcal{K}(D)_0]{} \mathcal{K}(X)_1 \xleftarrow[\mathcal{K}(D)_1]{} \cdots$$

with

$$\mathcal{K}(X)_0 := \mathcal{K}(X) \qquad \mathcal{K}(D)_0 := \mathcal{K}(D)$$

$$\mathcal{K}(X)_{i+1} := (\mathcal{K}(X)_i)^{m(D)} \qquad \mathcal{K}(D)_{i+1} := \Pi(\mathcal{K}(D)_i^{m(D)})$$

so that

$$\mathcal{K}(X)_i \xleftarrow[\mathcal{K}(D)_i]{} \mathcal{K}(X)_{i+1}.$$

By the functoriality of $\mathcal{K}$ we have

$$\mathcal{K}^2(X) = \mathcal{K}(\mathcal{K}(X)_0) \xleftarrow[\mathcal{K}(\mathcal{K}(D)_0)]{} \mathcal{K}(\mathcal{K}(X)_1)^{m(\mathcal{K}(D)_0)}$$

and

$$\mathcal{K}(\mathcal{K}(X)_1) \xleftarrow[\mathcal{K}(\mathcal{K}(D)_1)]{} \mathcal{K}(\mathcal{K}(X)_2)^{m(\mathcal{K}(D)_1)}.$$

Hence, by the definition of the product

$$\mathcal{K}(\mathcal{K}(X)_1)^{m(\mathcal{K}(D)_0)} \xleftarrow[\Pi(\mathcal{K}(\mathcal{K}(D)_1)^{m(\mathcal{K}(D)_0)})]{} (\mathcal{K}(\mathcal{K}(X)_2)^{m(\mathcal{K}(D)_0)})^{m(\mathcal{K}(D)_1)}.$$

Again we obtain a co-chain

$$\mathcal{K}^2(X)_0 := \mathcal{K}^2 \qquad \mathcal{K}^2(D_0) := \mathcal{K}^2(D)$$

$$\mathcal{K}_1^{2(X)} := \mathcal{K}(\mathcal{K}(X)_1)^{m(\mathcal{K}(D)_0)} \quad \mathcal{K}^2(D)_1 := \Pi(\mathcal{K}(\mathcal{K}(D)_1)^{m(\mathcal{K}(D)_0)})$$

$$\mathcal{K}^2(X)_{i+1} := (\cdots(\mathcal{K}(\mathcal{K}(X)_{i+1})^{m(\mathcal{K}(D)_0)})\ldots)^{m(\mathcal{K}(D)_i)}$$

$$\mathcal{K}^2(D)_{i+1} := \Pi(\Pi(\cdots\Pi(\mathcal{K}(\mathcal{K}(D)_{i+1})^{m(\mathcal{K}(D)_0)})\ldots)^{m(\mathcal{K}(D)_i)})$$

Definition

A family $(X_i, D_i)_{i\in\mathbb{N}}$ is a co-chain structure if for all $i \in \mathbb{N}$, X_i is a compact metric space and D_i a finite set of contractions $X_{i+1} \to X_i$, that is

$$X_i \underset{D_i}{\longleftarrow} X_{i+1}.$$

Let $((X_i, \rho_i), D_i)_{i\in\mathbb{N}}$ be a co-chain structure of covering extended IFS. Set

$$Z := \bigcup_{i\in\mathbb{N}} \{i\} \times X_i, \qquad \mathcal{D}(Z) := \bigcup_{i\in\mathbb{N}} \{i\} \times D_i.$$

Define

$$\rho((i,x),(j,y)) := \begin{cases} \rho_i(x,y) & \text{if } i = j, \\ \infty & \text{otherwise.} \end{cases}$$

Then ρ is an ∞-metric on Z.

Moreover, for $(i, d) \in \mathcal{D}(Z)$ and $(j, x) \in Z$ set

$$(i, d)(j, x) := (i, d(x))$$

if $j = i + 1$. Otherwise, let (i, d) be undefined. Then $(Z, \mathcal{D}(Z))$ is a covering generalised IFS, that is

$$Z = \bigcup_{(i,d)\in\mathcal{D}(Z)} \mathrm{range}((i, d)).$$

Let $\mathbb{C}_Z$ be the co-inductively defined largest set such that for $(i, x) \in Z$,

$$(i, x) \in \mathbb{C}_Z \to (\exists (i, d) \in \mathcal{D}(Z))(\exists (i{+}1, y) \in Z)\, (i, x) = (i, d)(i{+}1, y).$$

Then (classically)

$$\mathbb{C}_Z = Z.$$

Set $\quad \mathbb{C}_{X_0} := \{\, x \mid (0, x) \in \mathbb{C}_Z \,\}.$

6. Morphisms

Let $(X_i, D_i)_{i\in\mathbb{N}}, (Y_i, E_i)_{i\in\mathbb{N}}$ be co-chain structures and

$$X := \bigcup_{i\in\mathbb{N}} \{i\} \times X_i, \quad Y := \bigcup_{i\in\mathbb{N}} \{i\} \times Y_i,\ E \ = \bigcup_{i\in\mathbb{N}} \{i\} \times E_i.$$

Moreover, for $m > 0$, $j \in \mathbb{N}$, and $j_1 \leqslant \cdots \leqslant j_m \in \mathbb{N}$ let

$$\mathbb{F}(X, Y)^{(j)}_{j_1,\dots,j_m} := \{\, f : X^m \rightharpoonup Y \mid \mathrm{dom}(f) = \bigtimes_{\nu=1}^{m} \{j_\nu\} \times X_{j_\nu} \wedge \mathrm{range}(f) \subseteq \{j\} \times Y_j \,\},$$

$$\mathbb{F}(X, Y)_{j_1,\dots,j_m} := \bigcup \{\, \mathbb{F}(X, Y)^{(j)}_{j_1,\dots,j_m} \mid j \in \mathbb{N} \,\},$$

$$\mathbb{F}(X, Y)^{(j)} := \{\, \mathbb{F}(X, Y)^{(j)}_{j_1,\dots,j_m} \mid j_1 \leqslant \cdots \leqslant j_m \in \mathbb{N} \,\},$$

$$\mathbb{F}(X, Y) := \bigcup_{m>0, j\in\mathbb{N}} \ \bigcup_{j_1 \leqslant \cdots j_m} \mathbb{F}(X, Y)^{(j)}_{j_1,\dots,j_m}.$$

The following is a generalisation of U. Berger's co-inductive-inductive characterisation of the uniformly continuous functions on the unit interval.

Define $\Phi : \mathcal{P}(\mathbb{F}(X, Y)) \to (\mathcal{P}(\mathbb{F}(X, Y)) \to \mathcal{P}(\mathbb{F}(X, Y)))$ by

$$\begin{aligned}\Phi(F)(G) := \{\, f \in \mathbb{F}(X, Y) \mid \\ & [(\exists (i, e) \in E)(\exists h \in F \cap \mathbb{F}(X, Y)^{(i+1)})\, f = (i, e) \circ h] \vee \\ & [(\exists j_1 \leqslant \cdots \leqslant j_{\mathrm{ar}(f)} \in \mathbb{N})\, f \in \mathbb{F}(X, Y)_{j_1,\ldots,j_{\mathrm{ar}(f)}} \wedge \\ & \quad (\exists 1 \leqslant \nu \leqslant \mathrm{ar}(f))(\forall d \in D_{j_\nu}) f \circ (j_\nu, d^{(\nu,\mathrm{ar}(f))}) \in G] \,\}\end{aligned}$$

where

$$\begin{aligned} & d^{(\nu,m)}((j_1, x_1), \ldots, (j_m, x_m)) := \\ & \quad ((j_1, x_1), \ldots, (j_{\nu-1}, x_{\nu-1}), (j_\nu, d(x_\nu)), (j_{\nu+1}, x_{\nu+1}), \ldots, (j_m, x_m)),\end{aligned}$$

for $x_\kappa \in X_{j_\kappa}$ $(\kappa \in \{j_1, \ldots, j_m\} \backslash \{j_\nu\})$ and $x_\nu \in X_{j_\nu+1}$.

Set

$$\mathcal{J}(F) := \mu\Phi(F)(G).$$

Then $\mathcal{J}(F)$ is the least subset G of $\mathbb{F}(X, Y)$ so that

W If $(i, e) \in E$ and $h \in F \cap \mathbb{F}(X, Y)^{(i)}$ then $(i, e) \circ h \in G$.

R If $f \in \mathbb{F}(X, Y)$ and $\nu, j_1, \ldots, j_{\mathrm{ar}(f)} \in \mathbb{N}$ so that

- $j_1 \leqslant \cdots \leqslant j_{\mathrm{ar}(f)}$ and $f \in \mathbb{F}(X, Y)^{\nu}_{j_1,\ldots,j_{\mathrm{ar}(f)}}$
- $1 \leqslant \nu \leqslant \mathrm{ar}(f)$ and for all $d \in D_{j_\nu}$, $f \circ d^{(\nu,\mathrm{ar}(f))} \in G$,

then $f \in G$.

Set

$$\mathbb{C}_{\mathbb{F}(X,Y)} := \nu\mathcal{J} \quad \text{and} \quad \mathbb{C}_{\mathbb{F}(X_0,Y_0)} := \mathbb{C}_{\mathbb{F}(X,Y)} \cap \bigcup_{m>0} \mathbb{F}(X, Y)^{(0)}_{0^{(m)}}$$

where $x^{(m)} := (x, \ldots, x)$ (m times).

7. Category

- Objects: $\mathbb{C}_{\mathcal{K}^n(X)_0}$, for compact metric IFS (X, D), $n \in \mathbb{N}$.
- Morphisms: $\mathrm{Hom}(\mathbb{C}_{\mathcal{K}^n(X)_0}, \mathbb{C}_{\mathcal{K}^m(Y)_0}) := \mathbb{C}_{\mathbb{F}(\mathbb{C}_{\mathcal{K}^n(X)_0}, \mathbb{C}_{\mathcal{K}^m(Y)_0})}$

For a co-chain structure $X = (X_i, D_i)_{i \in \mathbb{N}}$ set

$$\eta_X(0, x) := (0, \{x\})$$
$$\eta_X(i+1, x) := (i+1, (\cdots((x^{(m(D_0))})^{(m(D_1))})\cdots)^{(m(D_i))})$$

Then $\eta_X(i, x) \in \{i\} \times \mathcal{K}(X_i)$. Moreover,

$$\eta_X \in \mathbb{C}_{\mathbb{F}(X, \mathcal{K}(X))}.$$

where $\mathcal{K}(X) := \bigcup_{i \in \mathbb{N}} \{i\} \times \mathcal{K}(X_i)$.

In addition, define

$$U_X(0,\mathbb{K}) := (0, \bigcup \mathbb{K})$$

$$U_X(i+1,\mathbb{K}) := (i+1, \langle \cdots \langle \langle \bigcup^{(m(D_0))} \rangle^{(m(D_1))} \rangle \cdots \rangle^{(m(D_i))} (\mathbb{K})).$$

Then $U \in \mathbb{C}_{\mathbb{F}(\mathcal{K}^2(X),\mathcal{K}(X))}$.