First Order Expansion of Intuitionistic Epistemic Logic

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23th March, MLA 2020/2021

Bivalence principle

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If a subsequence 12345 appears in somewhere in this sequence?

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Let p stand for case “12345 appears in this sequence”.

- In the classical logic, according to the bivalence principle, the truth value of p is either true or false.

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Question

If a subsequence 12345 appears in somewhere in this sequence?

Let *p* stand for case “12345 appears in this sequence”.

- In the classical logic, according to the bivalence principle, the truth value of *p* is either true or false.
- In the intuitionistic logic, we need to check every number appearing in the infinite sequence, at least there must be a algorithm to do so.

Law of excluded middle

Law of excluded middle holds in the classical logic but does not in the intuitionistic logic.

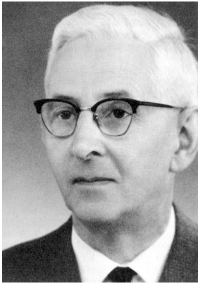
In the classical logic

The truth value of $p \vee \neg p$ is true, if the truth value of p or $\neg p$ is true.

In the intuitionistic logic

A proof of $p \vee \neg p$ consists in a proof of p or a proof of $\neg p$.

Brouwer-Heyting-Kolmogorov Interpretation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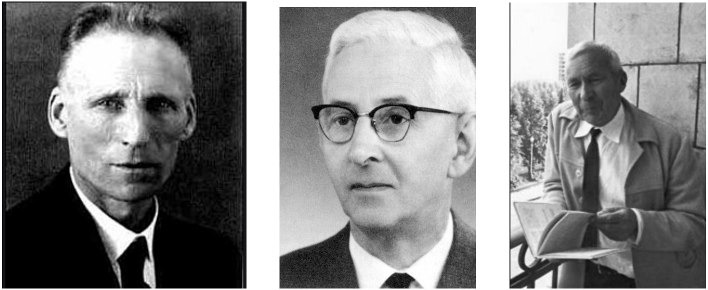
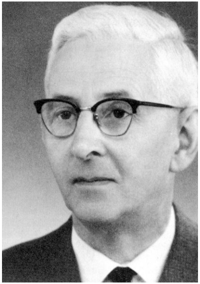


- A proof of $A\&B$ consists in
  a proof of A and a proof of B;

Brouwer-Heyting-Kolmogorov Interpretation


- A proof of $A\&B$ consists in
  a proof of A and a proof of B;
- A proof of $A \supset B$ consists in a construction
  which given a proof of A returns a proof of B.

(cf. van Dalen & Troelstra (1988))

Intuitionistic Epistemic Logic (Artemov et al. (2016))

Classic Epistemic Logic

KA is read as A is known as the case.

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A proof of a formula KA is the conclusive verification of the existence of a proof of A.

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- Example of Conclusive Verification: In the classical or intuitionistic **FOL**, an argument for sentence A being semantically valid is the conclusive verification of the existence of a proof of A.

Validity of $A \supset KA$ (Coreflection)

Recall that

- A proof of $A \supset B$ consists in a construction which given a proof of A returns a proof of B.
- A proof of KA is the conclusive verification of the existence of a proof of A.

Validity of $A \supset KA$ (Coreflection)

Recall that

- A proof of $A \supset B$ consists in a construction which given a proof of A returns a proof of B.
- A proof of KA is the conclusive verification of the existence of a proof of A.

BHK interpretation for $A \supset KA$

A proof of $A \supset KA$ consists in a construction which given a proof of A returns a conclusive verification of the existence of a proof of A.

Furthermore, this BHK-interpretation also trivializes the Knowability paradox.

Invalidity of $KA \supset A$

BHK interpretation for T-axiom : $KA \supset A$

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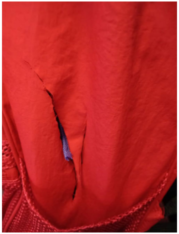
A proof of $KA \supset A$ consists in a construction which given a conclusive verification of the existence of a proof of A returns a proof of A.

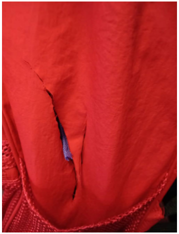
$KA \supset A$ is not valid, because:

- An argument for the validity of a sentence A does not directly give us a construction of a proof of A.

A Stolen Wallet



A Stolen Wallet



Artemov and Protopopescu (2016) gave an example:

Somebody stole your wallet in the subway. You have all evidence for this: the wallet is gone, your backpack has a cut in the corresponding pocket, but you have no idea who did it. You definitely know that "there is a person who stole my wallet".

BHK-interpretations for a Stolen Wallet

Let $S(x)$ stands for 'x stole my wallet'.

- a proof of $\exists x S(x)$ is the proof of $S(d)$ and the provision $d \in D$.
- a proof of $K(\exists x S(x))$ is the conclusive verification of the existence of a proof of $S(d)$ and the provision $d \in D$.

The latter requires a witness which is not satisfied. As a result, $K(\exists x S(x)) \supset \exists x S(x)$ does not hold intuitionistically.

However, Artemov and Protopopescu did not give a first-order version of **IEL**.

The Reasons for a First Order Expansion

Artemov and Protopopescu suggested that the notion of the intuitionistic knowledge captures both mathematical knowledge and empirical knowledge.

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Artemov and Protopopescu suggested that the notion of the intuitionistic knowledge captures both mathematical knowledge and empirical knowledge.

- When we consider the mathematical knowledge, quantifiers become inevitable.
- When we consider the empirical knowledge, we recall that Hintikka (1962) had mentioned that if we want to deal with the locutions like “knows who,” “knows when,” “knows where,” we need a language with quantifiers.

Our Goals: First-Order Expansion **IEL**

We name the first-order expansion of **IEL** as **QIEL**.

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Goal 3

Prove semantic completeness theorem of **QIEL**.

Outline

1. Hilbert System of **QIEL** and Its Semantics
2. Cut-free Analytic Sequent Calculus of **QIEL**
3. Semantic Completeness Theorem of **QIEL**

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Goal 1

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Propose a Hilbert system of **QIEL** and its semantics.

Syntax

A term t and a formula A of $\mathcal{L}(\mathcal{C} \cup \mathcal{V})$ is defined inductively.

$$t ::= x|c|f(\overrightarrow{t}).$$

$$A ::= P(\overrightarrow{t}) \,|\bot\, |A \wedge A\,|A \vee A\,|A \supset A\,|\forall xA\,|\exists xA\,|KA,$$

where $\overrightarrow{t}$ is $t_1, ..., t_n$ for $n \in \mathbb{N}$ and P denotes the predicate symbols. And $\neg A$ is defined as $A \supset \bot$.

Hilbert system of **QIEL**: Intuitionistic Part

($\wedge$-Ax)	$A_1 \wedge A_2 \supset A_i$ $(i = 1 \text{ or } i = 2)$
	$A \supset (B \supset (A \wedge B))$
($\vee$-Ax)	$A_i \supset A_1 \vee A_2$ $(i = 1 \text{ or } i = 2)$
	$(A \supset C) \supset ((B \supset C) \supset (A \vee B \supset C))$
($\supset$-Ax)	$A \supset (B \supset A)$
	$(A \supset (B \supset C)) \supset ((A \supset B) \supset (A \supset C))$
($\bot$-Ax)	$\bot \supset A$
($\forall$-Ax)	$\forall x A \supset A(t/x)$
($\exists$-Ax)	$A(t/x) \supset \exists x A$

(MP)	From A and $A \supset B$, infer B.
($\forall$-Rule)	From $A \supset B(u/x)$, infer $A \supset \forall x B$†
($\exists$-Rule)	From $B(u/x) \supset A$, infer $\exists x B \supset A$†

Hilbert system of **QIEL**: Epistemic Part

($\wedge$-Ax)	$A_1 \wedge A_2 \supset A_i$ ($i = 1$ or $i = 2$)
	$A \supset (B \supset (A \wedge B))$
($\vee$-Ax)	$A_i \supset A_1 \vee A_2$ ($i = 1$ or $i = 2$)
	$(A \supset C) \supset ((B \supset C) \supset (A \vee B \supset C))$
($\supset$-Ax)	$A \supset (B \supset A)$
	$(A \supset (B \supset C)) \supset ((A \supset B) \supset (A \supset C))$
($\bot$-Ax)	$\bot \supset A$
($\forall$-Ax)	$\forall x A \supset A(t/x)$
($\exists$-Ax)	$A(t/x) \supset \exists x A$
(K)	$K(A \supset B) \supset (KA \supset KB)$
(CR)	$A \supset KA$
(IR)	$KA \supset \neg\neg A$
(MP)	From A and $A \supset B$, infer B.
($\forall$-Rule)	From $A \supset B(u/x)$, infer $A \supset \forall x B\dagger$
($\exists$-Rule)	From $B(u/x) \supset A$, infer $\exists x B \supset A\dagger$

Validity of $KA \supset \neg\neg A$

Since $\vdash KA \supset \neg\neg A \equiv (\neg A \& KA) \supset \bot$, we consider the latter.

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BHK interpretation for $(KA \& \neg A) \supset \bot$

A proof of $(\neg A \& KA) \supset \bot$ consists in a construction which given a proof of $\neg A$ and the conclusive verification of the existence of a proof of A returns a proof of contradiction.

Furthermore, $\vdash KA \supset \neg\neg A \equiv \neg K\bot$.

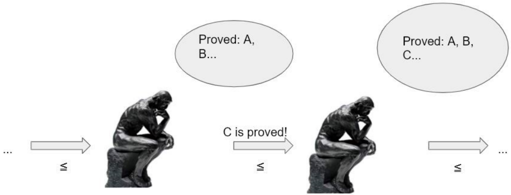
Kripke Semantics for IEL

A model is a tuple $M = (W, \leq, R, D, I)$ where

- $(W, \leq)$ is a preorder.
- R is a binary relation on W s.t.
  1. $R \subseteq \leq$ $\quad$ ($\because$) validity of $A \supset KA$
  2. $\leq; R \subseteq R$
  3. R is serial $\quad$ ($\because$) validity of $KA \supset \neg\neg A$
- $D = (D(w))_{w \in W}$ s.t. $w \leq v$ implies $D(w) \subseteq D(v)$.
- I is an interpretation s.t.
  1. $I(c) \in D(w)$ for all $w \in W$
  2. $I(P, w) \subseteq D(w)^n$ for all $w \in W$ s.t. if $u \leq v$ then $I(P, u) \subseteq I(P, v)$.
  3. $I(f, w) : D(w)^m \to D(w)$ s.t. if $w \leq v$ then for any $u_1, ..., u_n \in D(w)$, $I(f, w)(u_1, ..., u_n) = I(f, v)(u_1, ..., u_n)$.

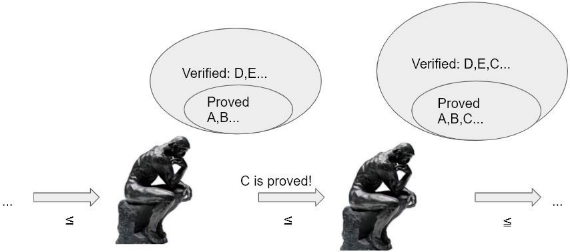
Connection of Semantics with BHK Int.

Recall Brouwer's notion of idealized mathematician: he extends his proved propositions and his universe of objects in the course of time. (cf. van Dalen 2004)

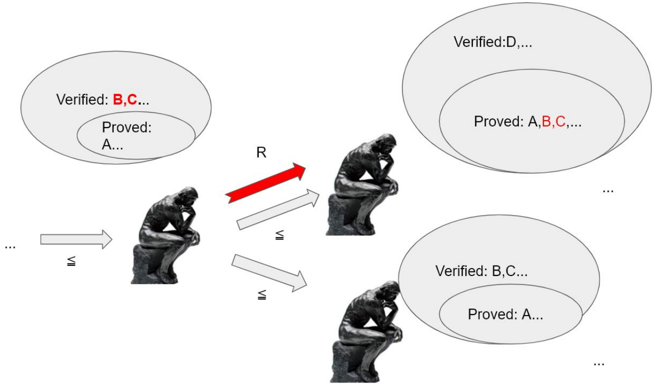


$w \leq v$ implies $\text{Proved}_w \subseteq \text{Proved}_v$.

Verification of the Idealized Mathematician



- To handle KA, we introduce the notion of verified propositions: KA is proved at w iff A is verified at w.
- So, we have a pair of proved and verified propositions (Proved_w, Verified_w) at each time w.
- $\text{Proved}_w \subseteq \text{Verified}_w$ ($\because A \supset KA$)



- Let us define R by;

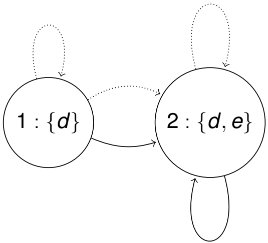
$$wRv \text{ iff } w \leq v \text{ and } \text{Verified}_w \subseteq \text{Proved}_v.$$

- All the required conditions for R are satisfied.

An Example of a Model

- $W = \{1, 2\}$, $D(1) = \{d\}$, $D(2) = \{d, e\}$.
- $I(S, 1) = \emptyset$, $I(S, 2) = \{d\}$.

Dotted lines stand for $\leq$ and solid lines stand for R.



- Both $R \subseteq \leq$ and $\leq; R \subseteq R$ hold and R is serial.

Truth Definition

Given a model $M = (W, \leq, R, D, I)$, a state $w \in W$ and a closed formula A and term t in $\mathcal{L}(D(w))$ which is expanded with the names of the elements in w. We define $I(t, w)$ as follows:

1. if t is a constant symbol c, $I(t, w) = I(c)$;
2. if t is in the form of $f(t_1, ..., t_n)$, $I(t, w) = I(f, w)(I(t_1, w), ..., I(t_n, w))$.

Truth Definition (Cont.)

The satisfaction relation $M, w \models A$ is defined as:

$$M, w \models P(t_1, \dots, t_m) \iff (I(t_1, w), \dots, I(t_m, w)) \in I(P, w)$$

$$M, w \models KA \iff \text{For all } v \in W, wRv \text{ implies } M, v \models A$$

$$M, w \models \forall xA \iff \text{For all } v \in W, w \leq v \text{ implies } M, v \models A(\underline{d}/x) \text{ For all } d \in D(v)$$

$$M, w \models \exists xA \iff M, w \models A(\underline{d}/x) \text{ for some } d \in D(w)$$

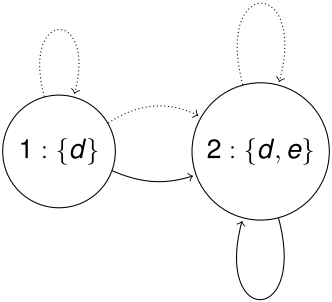
where $\underline{d}$ is the name of d.

Proposition (Persistency)

Let M be a model, for any $w, v \in W$, for any closed formulas A if $w \leq v$ and $M, w \models A$ then $M, v \models A$.

- $I(S,1) = \emptyset$, $I(S,2) = \{d\}$.

Dotted lines stand for $\leq$ and solid lines stand for R.



$M, 1 \not\models \exists x S(x)$

- $I(S,1) = \emptyset$, $I(S,2) = \{d\}$.

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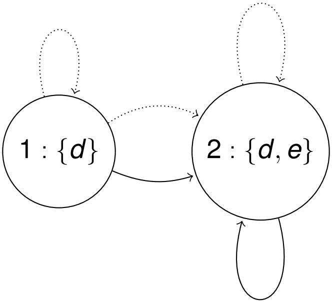

$M, 1 \not\models \exists x S(x)$ $\quad$ $M, 2 \models \exists x S(x)$

- $I(S,1) = \emptyset$, $I(S,2) = \{d\}$.

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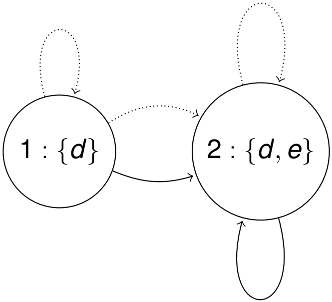

$M, 1 \not\models \exists x S(x)$ $\qquad$ $M, 2 \models \exists x S(x)$

$M, 1 \models K\exists x S(x)$

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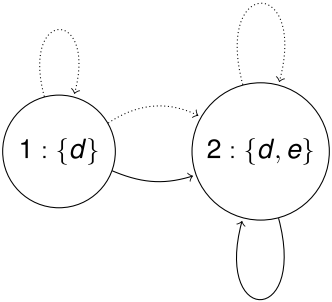
Dotted lines stand for $\leq$ and solid lines stand for R.



$M, 1 \not\models \exists x S(x)$ $\qquad$ $M, 2 \models \exists x S(x)$

$M, 1 \models K\exists x S(x)$

$M, 1 \not\models K\exists x S(x) \supset \exists x S(x)$

Soundness of $\mathcal{H}$(**QIEL**)

$\models A$ means that $M, w \models A$ for all models M and all w in M.

Theorem (Soundness of $\mathcal{H}$(QIEL))

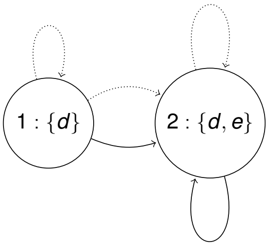
For any formula A, if $\mathcal{H}(\textbf{QIEL}) \vdash A$ then $\models A$.

A Counter Model for the “Stolen Wallet”

Proposition

$\mathcal{H}(\mathbf{QIEL}) \nvdash K(\exists x S(x)) \supset \exists x S(x)$.

- $I(S, 1) = \emptyset$, $I(S, 2) = \{d\}$.



$$M, 1 \not\models K\exists x S(x) \supset \exists x S(x).$$

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1 Hilbert System of **QIEL** and Its Semantics

2 Cut-free Analytic Sequent Calculus of **QIEL**

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Goal 2

Propose a cut-free analytic sequent calculus of **QIEL**.

Sequent

Let Γ be a finite multiset of formulas.

- A sequent $\Gamma \Rightarrow A$ can be read as

  "if all formulas in Γ hold then A holds."

- A sequent $\Gamma \Rightarrow$ can be read as

  "it cannot be the case that all formulas in Γ hold."

Sequent Calculus LJ

Δ contains at most one formula below.

$$A \Rightarrow A \qquad \bot \Rightarrow$$

$$\frac{\Gamma \Rightarrow \Delta}{A, \Gamma \Rightarrow \Delta}\,(LW) \quad \frac{\Gamma \Rightarrow}{\Gamma \Rightarrow C}\,(RW) \quad \frac{A, A, \Gamma \Rightarrow \Delta}{A, \Gamma \Rightarrow \Delta}\,(LC)$$

$$\frac{\Gamma \Rightarrow A \qquad A, \Gamma' \Rightarrow \Delta}{\Gamma, \Gamma' \Rightarrow \Delta}\,(Cut)$$

Sequent Calculus LJ

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$$A \Rightarrow A \qquad \bot \Rightarrow$$

$$\frac{\Gamma \Rightarrow \Delta}{A, \Gamma \Rightarrow \Delta}(LW) \quad \frac{\Gamma \Rightarrow}{\Gamma \Rightarrow C}(RW) \quad \frac{A, A, \Gamma \Rightarrow \Delta}{A, \Gamma \Rightarrow \Delta}(LC)$$

$$\frac{\Gamma \Rightarrow A \qquad A, \Gamma' \Rightarrow \Delta}{\Gamma, \Gamma' \Rightarrow \Delta}(Cut)$$

$$\frac{\Gamma \Rightarrow A_1 \qquad \Gamma \Rightarrow A_2}{\Gamma \Rightarrow A_1 \& A_2}(R\&) \quad \frac{A_i, \Gamma \Rightarrow \Delta}{A_1 \& A_2, \Gamma \Rightarrow \Delta}(L\&)$$

$$\frac{\Gamma \Rightarrow A_i}{\Gamma \Rightarrow A_1 \vee A_2}(R\vee) \quad \frac{A_1, \Gamma \Rightarrow \Delta \qquad A_2, \Gamma \Rightarrow \Delta}{A_1 \vee A_2, \Gamma \Rightarrow \Delta}(L\vee)$$

$$\frac{A, \Gamma \Rightarrow B}{\Gamma \Rightarrow A \supset B}(R\supset) \quad \frac{\Gamma \Rightarrow A \qquad B, \Gamma' \Rightarrow \Delta}{A \supset B, \Gamma, \Gamma' \Rightarrow \Delta}(L\supset)$$

Sequent Calculus LJ

$$\frac{\Gamma \Rightarrow A(u/x)}{\Gamma \Rightarrow \forall x A}\,(R\forall)\dagger \quad \frac{A(t/x), \Gamma \Rightarrow \Delta}{\forall x A, \Gamma \Rightarrow \Delta}\,(L\forall)$$

$$\frac{\Gamma \Rightarrow A(t/x)}{\Gamma \Rightarrow \exists x A}\,(R\exists) \quad \frac{A(u/x), \Gamma \Rightarrow \Delta}{\exists x A, \Gamma \Rightarrow \Delta}\,(L\exists)\dagger$$

†: u does not occur in the lower sequent.

Sequent Calculus $\mathcal{G}$(**QIEL**)

To define G1-style $\mathcal{G}$(**QIEL**), add the following rule (K_{IEL}) to **LJ**:

$$\frac{\Gamma_1, \Gamma_2 \Rightarrow \Delta}{\Gamma_1, K\Gamma_2 \Rightarrow K\Delta}\ (K_{IEL})$$

where Δ contains at most one formula, so possibly empty.

- (K_{IEL}) satisfies the subformula property.

Sequent Calculus by Krupski, et al. (2016)

They gave sequent calculus of propositional **IEL** and added (KI) and (U) to the propositional part of **LJ**:

$$\frac{\Gamma_1, \Gamma_2 \Rightarrow A}{\Gamma_1, K\Gamma_2 \Rightarrow KA}\ (KI) \qquad \frac{\Gamma \Rightarrow K\bot}{\Gamma \Rightarrow F.}\ (U)$$

Sequent Calculus by Krupski, et al. (2016)

They gave sequent calculus of propositional **IEL** and added (KI) and (U) to the propositional part of **LJ**:

$$\frac{\Gamma_1, \Gamma_2 \Rightarrow A}{\Gamma_1, K\Gamma_2 \Rightarrow KA}\ (KI) \qquad \frac{\Gamma \Rightarrow K\bot}{\Gamma \Rightarrow F.}\ (U)$$

- However, (U) does not enjoy the subformula property.
- Two systems are equivalent for the propositonal part.

Equivalence of $\mathcal{H}$(**QIEL**) and $\mathcal{G}$(**QIEL**)

Theorem

$\mathcal{H}(\textbf{QIEL}) \vdash A$ iff $\mathcal{G}(\textbf{QIEL}) \vdash\Rightarrow A$.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \Rightarrow A \qquad \bot \Rightarrow}{A, \neg A \Rightarrow}\,(L\supset)}{KA, \neg A \Rightarrow}\,(K_{IEL})}{KA, \neg A \Rightarrow \bot}\,(RW)}{KA \Rightarrow \neg\neg A}\,(R\supset)}{\Rightarrow KA \supset \neg\neg A}\,(R\supset)}{}$$

How to Prove Cut-elimination

Let $\mathcal{G}^{-c}(\mathbf{IEL})$ be the system $\mathcal{G}(\mathbf{IEL})$ w/ (*Cut*) rule.

Theorem(Cut-elimination Theorem)

If $\mathcal{G}(\mathbf{IEL}) \vdash \Gamma \Rightarrow \Delta$ then $\mathcal{G}^{-c}(\mathbf{IEL}) \vdash \Gamma \Rightarrow \Delta$.

Our cut-elimination theorem is shown by a syntactic method from Ono (1985) and Kashima (2009). By this method, we eliminate the extended cut of the following form.

$$\frac{\Gamma \Rightarrow A \qquad A^n, \Gamma' \Rightarrow \Delta}{\Gamma, \Gamma' \Rightarrow \Delta} \; (Ecut)$$

where A^n means n-times repetition of the formula A and $n \geq 0$.

As a corollary of the cut-elimination theorem, we obtain:

Disjunction Property

If $\mathcal{G}(\mathbf{QIEL}) \vdash \Rightarrow A \vee B$,

then $\mathcal{G}(\mathbf{QIEL}) \vdash \Rightarrow A$ or $\mathcal{G}(\mathbf{QIEL}) \vdash \Rightarrow B$.

As a corollary of the cut-elimination theorem, we obtain:

Disjunction Property

If $\mathcal{G}(\textbf{QIEL}) \vdash \Rightarrow A \vee B$,

then $\mathcal{G}(\textbf{QIEL}) \vdash \Rightarrow A$ or $\mathcal{G}(\textbf{QIEL}) \vdash \Rightarrow B$.

Existence Property

If $\mathcal{G}(\textbf{QIEL}) \vdash \Rightarrow \exists x A$

then there exists a term t s.t. $\mathcal{G}(\textbf{QIEL}) \vdash \Rightarrow A(t/x)$.

Let $\mathrm{FV}(C)$, $\mathrm{PR}(C)$, $\mathrm{Fr}(C)$ and $\mathrm{Con}(C)$ denote the free variables, predicate symbols, function symbols and constant symbols in a formula C, respectively.

Craig Interpolation (by Maehara's method)

If $\mathcal{G}(\mathbf{QIEL}) \vdash \Rightarrow A \supset B$, then there exists a formula C such that $\mathcal{G}(\mathbf{QIEL}) \vdash \Rightarrow A \supset C$, $\mathcal{G}(\mathbf{QIEL}) \vdash \Rightarrow C \supset B$, and the following:

- $\mathrm{FV}(C) \subseteq \mathrm{FV}(A) \cap \mathrm{FV}(B)$
- $\mathrm{PR}(C) \subseteq \mathrm{PR}(A) \cap \mathrm{PR}(B)$
- $\mathrm{Fr}(C) \subseteq \mathrm{Fr}(A) \cap \mathrm{Fr}(B)$.
- $\mathrm{Con}(C) \subseteq \mathrm{Con}(A) \cap \mathrm{Con}(B)$.

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3. Semantic Completeness Theorem of **QIEL**

Completeness of Sequent Calculus

Goal 3

Prove semantic completeness theorem of **QIEL** .

This also gives us a semantic proof of cut-elimination.

Strategy to prove the Completeness

Definition

$\models \Gamma \Rightarrow \Delta$ means that $M, w \models \bigwedge \Gamma \supset \bigvee \Delta$ for all models M and all states w in M.

Let $\mathcal{G}^{-}(\mathbf{QIEL})$ be the system $\mathcal{G}(\mathbf{QIEL})$ w/ (Cut) rule.

Strategy to Prove Completeness

$\models \Gamma \Rightarrow \Delta \overset{(1)}{\Longrightarrow} \mathcal{G}^{-}(\mathbf{QIEL}) \vdash \Gamma \Rightarrow \Delta \overset{(2)}{\Longrightarrow} \mathcal{G}(\mathbf{QIEL}) \vdash \Gamma \Rightarrow \Delta$, where $|\Delta| \leq 1$.

(2) is obvious, and so we show (1) alone.

Cut-free Completeness

If $\models \Gamma \Rightarrow \Delta$ then $\mathcal{G}^{-}(\mathbf{QIEL}) \vdash \Gamma \Rightarrow \Delta$.

The proof can be carried in the setting of

- multi-succedent system (cf. Mints (2000))
- single-succedent system (cf. Hermant (2005))

We choose the second method.

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   - $\mathcal{G}^{-}(\textbf{QIEL}) \nvdash \Gamma^{+} \Rightarrow \Delta$;
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2. Then Γ can be extended to Γ^+ in an expanded language by countably infinite fresh variables, s.t. Γ^+ is Δ-saturated, i.e.,
   - $\mathcal{G}^-(\textbf{QIEL}) \nvdash \Gamma^+ \Rightarrow \Delta$;
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     for any B, either $\mathcal{G}^-(\textbf{QIEL}) \vdash \Theta \cup \{B\} \Rightarrow \Delta$ or $B \in \Theta$;
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   - $W := \{\Sigma : \Sigma \text{ is } \Pi\text{-saturated in } \mathcal{G}^-(\textbf{QIEL}) \text{ for some } \Pi\}$.

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   - $D(\Sigma)$ is the set of all terms in Σ.

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   - $\Sigma R\, \Omega$ iff $K^{-}(\Sigma) \cup \Sigma \subseteq \Omega$ where $K^{-}(\Sigma) = \{B | KB \in \Sigma\}$.

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   - $\Sigma R \, \Omega$ iff $K^{-}(\Sigma) \cup \Sigma \subseteq \Omega$ where $K^{-}(\Sigma) = \{B | KB \in \Sigma\}$.
4. It follows that $M^{\Lambda}, \Gamma^{+} \not\models \Gamma \Rightarrow \Delta$.

Strategy to Prove Completeness

$\models \Gamma \Rightarrow \Delta \overset{(1)}{\Longrightarrow} \mathcal{G}^{-}(\textbf{QIEL}) \vdash \Gamma \Rightarrow \Delta \overset{(2)}{\Longrightarrow} \mathcal{G}(\textbf{QIEL}) \vdash \Gamma \Rightarrow \Delta$, where $|\Delta| \leq 1$.

- We have proved (1) and recall that (2) is obvious.

Semantic Proof of Cut-elimination

Strategy

$\mathcal{G}(\textbf{QIEL}) \vdash \Gamma \Rightarrow \Delta \overset{(0)}{\Longrightarrow} \models \Gamma \Rightarrow \Delta \overset{(1)}{\Longrightarrow} \mathcal{G}^{-}(\textbf{QIEL}) \vdash \Gamma \Rightarrow \Delta$, where $|\Delta| \leq 1$.

- (0) is easy.
- (1) is already shown by cut-free completeness.

Conclusion

Corollary

The following are all equivalent.

1. $\models A$,
2. $\mathcal{G}^{-}(\mathbf{QIEL}) \vdash \Rightarrow A$,
3. $\mathcal{G}(\mathbf{QIEL}) \vdash \Rightarrow A$,
4. $\mathcal{H}(\mathbf{QIEL}) \vdash A$.

Conclusion

We give a first-order expansion as **QIEL** of **IEL**.

1. proposes a Hilbert system and its semantics of **QIEL**;
2. proposes a sequent calculus, in which subformula property is satisfied.
3. proves cut-elimination theorems by a syntactic method, which leads to the results of
   - disjunction property;
   - existence Property;
   - Craig interpolation theorem.
4. proves completeness theorem of the sequent calculus.

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1. What first-order classical epistemic logic can be embedded into **QIEL** by double negation translation?
2. Adding the equality symbol into **QIEL** can be an important direction.
3. Provide a **G3**-style sequent calculus.
4. How to directly prove the completeness of $\mathcal{H}$(**QIEL**)?

Knowability Paradox

The study of BHK-interpretation of KA from **IEL** also trivialize the Knowability Paradox (Fitch-Church Paradox).

Knowability Paradox (Fitch-Church Paradox)

- (KP) Knowability principle $A \supset \Diamond KA$. Every truth is knowable.
- (OP) Omniscience principle $A \supset KA$. Every truth is known.

(OP) is classically derivable from (KP).

Since the (KP) is suggested by Dummett's semantic anti-realism and the (OP) seems weird, this paradox is commonly recognized as a threat to anti-realism.

Intuitionistic Solution

Dummett (p.164, 1963) states:

> *It will be obvious anyone familiar with the elements of intuitionism that I have taken some of its basic features as a model for an anti-realists view.*

- The method of switching an underlying logical system to an intuitionistic one to avoid the paradox seems very natural.
- Then if we can accept the BHK-interpretation of KA from **IEL**, $A \supset KA$ can be accepted.