# Computable analysis and verified exact real computation in Coq

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# Computable analysis in Coq

- Incone is a library for doing computable analysis in the Coq proof assistant.
- Definitions closely follow those from computable analysis.
- The internal logic of Coq is constructive.
- Axioms can be added to allow e.g. classical reasoning.
- Coq is often used for classical program verification
  Program $\rightarrow$ Specification $\rightarrow$ Correctness Proof

# Real analysis in Coq

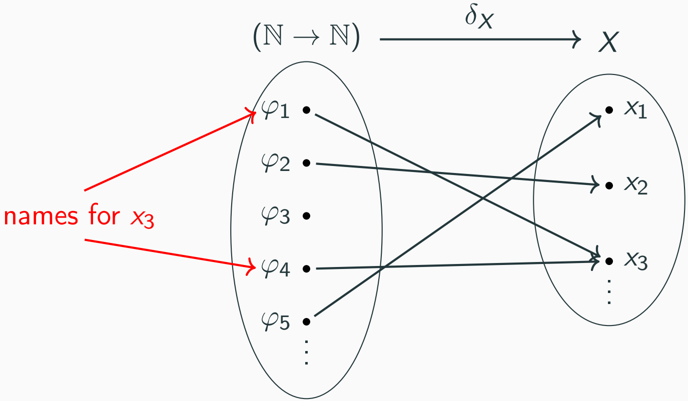
- The Coq standard library
  - Axiomatic definition of the reals
  - Not computational
  - Assumes existence of non-computable functions e.g. $up : \mathbb{R} \to \mathbb{N}$.
- CoRN
  - Completely constructive
  - Executable inside Coq

## The incone library

- The incone library formalizes results from computable analysis in Coq.
- Definitions closely follow those from computable analysis.
- Distinguishes between an algorithm and its specification.
- Correctness proofs can use classical mathematics (e.g. use the real numbers from the Coq standard library).
- Computability is not formalized and only reasoned about on the meta-level.

# Computable analysis and representations

# Representations




Represented space $\mathsf{X} := (X, \delta_X)$.

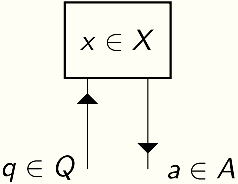
# Represented spaces

Computable analysis uses explicit encodings by natural numbers or finite binary strings.
However, here we consider more general encodings by "basic types".

## Represented space

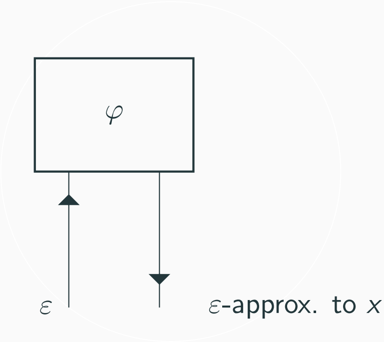
A represented space X consists of

- An abstract base type $X$.
- A space of names $\mathcal{B}_{\mathsf{X}} := Q \to A$ with questions $Q$ and answers $A$ and proofs that they are countable.
- A partial, surjective function $\delta :\subseteq \mathcal{B}_{\mathsf{X}} \to X$ called representation.



# Example: $\mathbb{R}_{\mathbb{Q}}$

A representation for the reals is given by choosing questions and answers to be $\mathbb{Q}$ and $\varphi : \mathbb{Q} \to \mathbb{Q}$ is a name for $x \in \mathbb{R}$ if $\forall \varepsilon > 0, |\varphi(\varepsilon) - x| \leq \varepsilon$.



# Cauchy Representation in Coq

Easy to define in Coq using the axiomatization of the reals in the standard library:
$\varphi$ is name for $x$: $\forall \varepsilon > 0, |x - \varphi(\varepsilon)| \leq \varepsilon$.

```
(* A name for x encodes x by rational approximations *)
Definition is_name (phi : (Q -> Q)) (x : R) :=
      forall eps, (0 < (Q2R eps)) ->
        Rabs (x - (phi eps)) <= eps.
```

Example: $\varepsilon \mapsto \varepsilon$ is a name for 0.

```
(* A name for zero *)
Lemma zero_name : (is_name (fun eps => eps) 0).
```

# Realizers

**Definition (Realizer)**

Let X, Y be represented spaces. $F :\subseteq \mathcal{B}_{\mathsf{X}} \to \mathcal{B}_{\mathsf{Y}}$ is a realizer for $f :\subseteq \mathsf{X} \to \mathsf{Y}$ if

$$\delta_{\mathsf{Y}} \circ F(\varphi) = f(x) \text{ for all } \varphi \in \delta_{\mathsf{X}}^{-1}(x) \text{ and } x \in X.$$

$$\begin{tikzcd} X \arrow[r, "f"] & Y \\ \mathcal{B}_{\mathsf{X}} \arrow[u, "\delta_{\mathsf{X}}"] \arrow[r, "F"'] & \mathcal{B}_{\mathsf{Y}} \arrow[u, "\delta_{\mathsf{Y}}"'] \end{tikzcd}$$

# Realizers for reals in coq

```
(* A realizer maps names to names *)
Definition is_realizer
         (F: (Q -> Q) -> Q -> Q) (f : R -> R) :=
           forall phi x, (is_name phi x) ->
               (is_name (F phi) (f x)).
```

- Realizers specify algorithms
- Correctness can be proven using classical mathematics

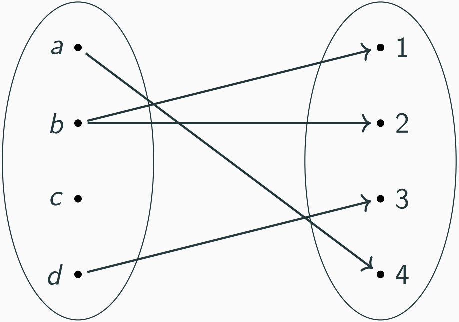
# Basic constructions on represented spaces

For represented spaces $\mathsf{X}$ and $\mathsf{Y}$ we can automatically define representations for

- The product space $\mathsf{X} \times \mathsf{Y}$.
- The co-product space $\mathsf{X} + \mathsf{Y}$.
- Infinite sequences $\mathsf{X}^\omega$.
- Spaces of subsets $A \subseteq \mathsf{X}$.
- The space of (continuous) functions $\mathsf{X} \to \mathsf{Y}$.

# Multivalued functions

A (partial) multivalued function $f :\subseteq X \rightrightarrows Y$ is just a relation or a set-valued function. The intuitive meaning is that several valid values exist.




A realizer for a multivalued function $f : X \rightrightarrows Y$ has to return a name for any $y \in f(x)$ when given a name for $x \in X$.

# Finite spaces and operations on multifunctions

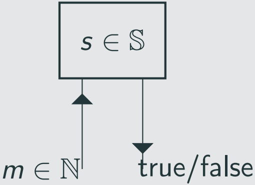
# Example: Sierpiński space

**Definition (Sierpiński space)**

Sierpiński space is the topological space with point set $\{\top, \bot\}$ and open sets $\emptyset$, $\{\top\}$ and $\{\top, \bot\}$.

**Example (Representation for Sierpiński space)**

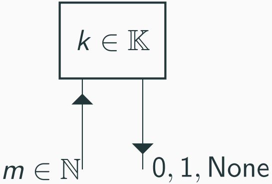
- Question space $Q_{\mathbb{S}} := \mathbb{N}$.
- Answer space $A_{\mathbb{S}} :=$ `bool`.
- $\delta_{\mathbb{S}}(\varphi) = \top \iff \exists n, \varphi(n) = \texttt{true}$.



# The Kleeneans

Kleene's three-valued logic is the logic on $\{0, 1, \perp\}$. It can be turned into a represented space as follows.

- Question space $Q_{\mathbb{K}} := \mathbb{N}$.
- Answer space $A_{\mathbb{K}} := \{0, 1, \text{None}\}$.
- Representation $\delta_{\mathbb{K}}$ such that $\delta_{\mathbb{K}}(\varphi)$ is the first value of $\varphi$ different from None or $\perp$ if no such value exists.
- Can define realizers for logical operations




The Kleeneans can be used to define a computable total real comparison $x <_{\mathbb{K}} y := \begin{cases} (x < y)_{\mathcal{B}} & \text{if } x \neq y \\ \perp_{\mathbb{K}} & \text{otherwise.} \end{cases}$

## Parallelization and Selection

For multifunctions $f : X \rightrightarrows Y$, $f' : X' \rightrightarrows Y'$ we can define $f + f' : X + X' \rightrightarrows Y + Y'$ and $f \times f' : X \times X' \rightrightarrows Y \times Y'$:

$$(f \times f')(x, x') := f(x) \times f'(x').$$

$$(f + f')(p) := \begin{cases} \operatorname{inl} f(x) & \text{if } p = \operatorname{inl} x \\ \operatorname{inr} f'(x') & \text{if } p = \operatorname{inr} x' \end{cases}$$

$f \times f'$ corresponds to running $f$ and $f'$ in parallel, $f + f'$ corresponds to selecting either $f$ or $f'$.

# Multivalued branching

Let $f, g : X \rightrightarrows Y$ and $b : X \rightrightarrows \mathbb{B}$ and consider the expression

$$\text{if } b(x) \text{ then } f(x) \text{ else } g(x).$$

Let $\mathsf{if}_{\mathsf{X}} : \mathbb{B} \times \mathsf{X} \to \mathsf{X} + \mathsf{X}$ be the function defined by

$$\mathsf{if}_{\mathsf{X}}(b, x) := \begin{cases} \mathsf{inl}\, x & \text{if } b = \mathrm{true} \\ \mathsf{inr}\, x & \text{if } b = \mathrm{false}. \end{cases}$$

Then the desired semantics can be expressed by

$$\nabla \circ (f + g) \circ \mathsf{if}_{\mathsf{X}} \circ (b \times \mathrm{id}) \circ \Delta(x).$$

# Exact real computation in Incone

# Verifying exact real computation in Incone

Typically writing and verifying an exact real computation algorithm in Incone consists of the following steps.

- Give an abstract mathematical description of the problem as a multifunction using e.g. the axiomatic real numbers from the Coq standard library.
- Define a representation for the real numbers.
- Define a (computable) function on the naming space.
- Prove that this function is a realizer for the multifunction.

## A structure for computable reals

In exact real computation, basic operations are combined to define more complicated programs. To do something similar in Coq, we define a structure computable reals containing the following basic operations

- Arithmetic operations
- The efficient limit $\lim_{\text{eff}} :\subseteq \mathbb{R}^\omega \to \mathbb{R}$, that maps any sequence $(x_i) \in \mathbb{R}^\omega$ that is efficiently Cauchy to its limit $\lim(x_i)$
- A Kleenean comparison function $<_\mathbb{K}$
- The function $\mathrm{FtoR} : \mathbb{Z} \times \mathbb{Z} \to \mathbb{R}$, $(m, e) \mapsto m \cdot 2^{-e}$
- Rational approximation $\mathrm{approx} : \mathbb{R} \times \mathbb{Q} \rightrightarrows \mathbb{Q}$

More complicated operations can be defined as composition of basic operations independently of the underlying representation.

# Interval Reals

Computing with rationals is not very efficient.
Alternative: approximate real numbers by intervals with dyadic endpoints (numbers of the form $m \cdot 2^e$).

### Definition

Let $\mathbb{ID}$ be the set of intervals with dyadic endpoints. A representation $\mathbb{R}_{\mathbb{ID}}$ of the reals is given by $Q_{\mathbb{ID}} = \mathbb{N}$, $A_{\mathbb{ID}} = \mathbb{ID}$.

$$\delta_{\mathbb{R}_{\mathbb{ID}}}((I_n)_{n\in\mathbb{N}}) = x \quad \iff \quad x \in \bigcap_{n\in\mathbb{N}} I_n \text{ and } \lim_{n\to\infty} |I_n| = 0.$$

The Coq Interval Library already provides many operations on intervals.

# Interval arithmetic in Coq

Correctness in interval arithmetic:

```
Lemma add_correct prec x y I J:
         x \contained_in I -> y \contained_in J ->
   (x + y) \contained_in (I.add prec I J).
```

To define a realizer we additionally need absolute error bounds to show that intervals get arbitrarily small:

```
Lemma add_error I J n m p x y N:
  diam I <= /2^n ->  diam J <= /2^m ->
  (x \contained_in I) -> (y \contained_in J) ->
  (Rabs x) <=  (2 ^ N) -> (Rabs y) <= (2 ^ N)
  -> diam (I.add p I J) <= /2 ^ n + /2 ^ m +
                          (2 ^ (N+5-p)).
```

# Square root

We give an efficient implementation of $\sqrt{x}$ for $x \in [0, \infty)$.
For this, we define a function $\texttt{sqrt\_approx} : \mathbb{R}^\omega \to \mathbb{R}$ with

$$\left|\texttt{sqrt\_approx}(x, n) - \sqrt{x}\right| \leq 2^{-n}$$

using only operations from the computable reals structure. Then

$$\texttt{sqrt} := \texttt{lim\_eff} \circ \texttt{sqrt\_approx}.$$

For $x \in [0.25, 2]$ the Heron iteration defined by

$$x_0 := 1$$
$$x_{n+1} := \frac{1}{2}\left(x_n + \frac{x}{x_n}\right)$$

converges quadratically, i.e.

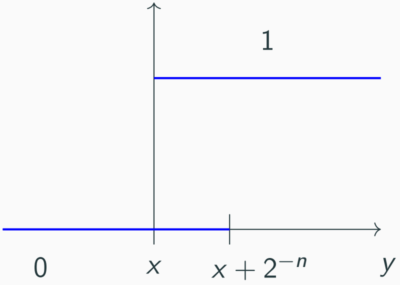
$$\texttt{sqrt\_approx}_{|[0.25,2]}(x, n) = x_{\log_2 n}$$

and then extend to all non-negative reals.

# Soft Comparison

Soft-comparison $sc\colon \mathbb{R} \times \mathbb{R} \times \mathbb{N} \rightrightarrows \mathbb{B}$ is defined by

$\mathrm{true} \in \mathrm{sc}(n, x, y) \Leftrightarrow x < y$ and $\mathrm{false} \in \mathrm{sc}(n, x, y) \Leftrightarrow y < x{+}2^{-n}$.

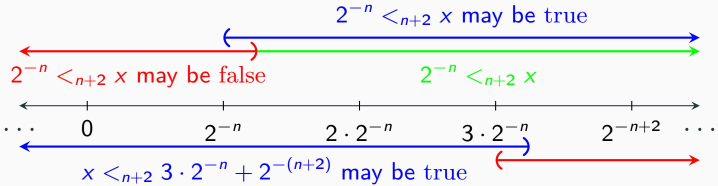



$\mathrm{sc}(n, x, y)$ plotted over $y$ for fixed $x$ and $n$.

# Magnitude and Scaling

We define a multifunction $\texttt{magnitude}: \mathbb{R} \rightrightarrows \mathbb{Z}$ such that

$$z \in \texttt{magnitude}(x) \Leftrightarrow 2^z < x < 2^{z+2}.$$



# Code extraction

For faster execution, Coq can extract Haskell or Ocaml code from proofs and definitions.
The basic extraction mainly performs a straightforward syntactic translation.
It can be improved by telling Coq how to extract:

```
Extract Inlined Constant Z.abs => "(Prelude.abs)".
Extract Inlined Constant Z.geb => "(Prelude.>=)".
Extract Inlined Constant Z.opp => "(Prelude.negate)".
Extract Inlined Constant Z.succ => "(Prelude.succ)".
Extract Inlined Constant Z.pow_pos => "(Prelude.^)".
Extract Inlined Constant Z.pow => "(Prelude.^)".
```

# Conclusion and Future work

- The incone library can be used to implement real number computations in Coq and do proofs in the style of computable analysis.
- Our simple implementation is already quite efficient.
- More complicated spaces and operators, e.g., analytic functions and ODE solving.

Thank you!