Takeuti's argument for the finitistic admissibility of transfinite induction

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Hilbert and Bernays, *Grundlagen der Mathematik* (1934)

Our treatment of the basics of number theory and algebra was meant to demonstrate how to apply and implement direct contentual inference that takes place in thought experiments [*Gedanken-Experimenten*] on intuitively conceived objects and is free of axiomatic assumptions. Let us call this kind of inference "finitist" inference for short, and likewise the methodological attitude underlying this kind of inference as the "finitist" attitude or the "finitist" standpoint.... With each use of the word "finitist", we convey the idea that the relevant consideration, assertion, or definition is confined to objects that are conceivable in principle, and processes that can be effectively executed in principle, and thus it remains within the scope of a concrete treatment.

Hilbert, "The Grounding of Elementary Number Theory" (1931)

This is the fundamental mode of thought that I hold to be necessary for mathematics and for all scientific thought, understanding, and communication, and without which mental activity is not possible at all.

Hilbert and Bernays, *Grundlagen der Mathematik* (1934)

Regarding this goal [of proving consistency], I would like to emphasize that an opinion, which had emerged intermittently—namely that some more recent results of Gödel would imply the infeasibility of my proof theory—has turned out to be erroneous. Indeed, that result shows only that—for more advanced consistency proofs—the finitist standpoint has to be exploited in a manner that is sharper [*schärferen*] than the one required for the treatment of the elementary formulations.

Tarski, "Contribution to the discussion of P. Bernays *Zur Beurteilung der Situation in der beweistheoretischen Forschung* (1954)

Gentzen's proof of the consistency of arithmetic is undoubtedly a very interesting metamathematical result, which may prove very stimulating and fruitful. I cannot say, however, that the consistency of arithmetic is now much more evident to me (at any rate, perhaps, to use the terminology of the differential calculus more evident than by an epsilon) than it was before the proof was given.

Girard, *The Blind Spot* (2011)

Concerning Gentzen's second consistency proof, André Weil said that "Gentzen proved the consistency of arithmetic, i.e., induction up to the ordinal ω, by means of induction up to ϵ_0", the venom being that ϵ_0 is much larger than ω.

Takeuti, "Consistency Proofs and Ordinals" (1975)

Anyway since I am a logician and am very familiar with the magic of quantifiers Gentzen's consistency proof, which consists of the elimination of quantifiers and an accessibility proof for the ordinals less than ϵ_0, is greatly reassuring. It does add to my confidence in the consistency and truth of Peano arithmetic.

Let's consider an outline of Gentzen's proof of the consistency of PA.

He designs a system of ordinals and an ordering of these ordinals that are each concrete and thus finitistically acceptable.

This ordering has type ϵ_0.

Proofs in PA are assigned these ordinals according to the rules of inference used.

He provides a procedure for reducing proofs so that each proof of inconsistency gets reduced to another proof of inconsistency with a smaller ordinal.

If there is a proof of inconsistency, this procedure generates an infinitely decreasing sequence of such ordinals.

By the well-ordering of the ordering of type ϵ_0, such a sequence is impossible.

Thus there is no proof of inconsistency in PA.

In Gentzen's proof every step except the well-ordering of the ordering of type ϵ_0 can be effected in primitive recursive arithmetic (generally accepted to be finitistically acceptable).

In particular, it needs to be proved that every strictly decreasing computable sequence of ordinals in this ordering is finite.

This is the part of the proof that needs to be justified from the finitist standpoint.

Takeuti, "Consistency Proofs and Ordinals" (1975)

Gödel's incompleteness theorem has changed the meaning of Hilbert's program completely. Because of Gödel's result consistency proofs now require a method that is finite (or constructive) but which is nevertheless very strong when formalized. People think this is impossible or at least unlikely and extremely difficult. The situation is somewhat similar to that of finding a new axiom that carries conviction and decides the continuum hypothesis.

The claim about decreasing sequences of ordinals has the provability strength of the consistency of PA, but is still, Takeuti alleges, finitistically acceptable.

Takeuti calls an ordinal μ accessible if it has been finitistically proved that every strictly decreasing sequence starting with μ is finite.

This is the step in Gentzen's proof that needs to be finitistically justified: that every ordinal up to ϵ_0 is accessible.

Takeuti observes that it is clear that every natural number is accessible.

The crux of his argument is to extend this observation to infinite ordinals.

Firstly, he argues that $\omega + \omega$ is accessible: the first term μ_0 of any decreasing sequence from $\omega + \omega$ is either of the form n or $\omega + n$.

If the former, then we're done.

If the latter, then consider the sequence $\mu_{n+1} < \cdots < \mu_2 < \mu_1 < \mu_0$.

This sequence has length $n + 2$ and thus μ_{n+1} must be a natural number.

Such reasoning will also show that ordinals to ω^ω are accessible.

Takeuti then introduces the notion of *n*-accessibility, defined inductively.

μ is 1-accessible if μ is accessible.

μ is $(n+1)$-accessible if for every *n*-accessible ν, $\nu \cdot \omega^\mu$ is *n*-accessible.

Takeuti, *Proof theory* (1975)
It should be emphasized that "μ being *n*-accessible" is a clear notion only when it has been concretely demonstrated that μ is *n*-accessible.

Lemma. If μ is *n*-accessible and $\nu < \mu$, then ν is *n*-accessible.

Lemma. Suppose $\{\mu_m\}$ is an increasing sequence of ordinals with limit μ. If each μ_m is *n*-accessible, then so is μ.

These will yield finite towers of ω ending in ω^μ, which can be seen to be *n*-accessible since μ is a limit ordinal of *n*-accessible ordinals.

Lemma. If ν is $(n+1)$-accessible, then so is $\nu \cdot \omega$.

We must show that for each n-accessible μ, $\mu \cdot \omega^{\nu \cdot \omega}$ is n-accessible.

Since ω is a limit ordinal, by the previous lemma it suffices to show that $\mu \cdot \omega^{\nu \cdot m}$ is n-accessible for all m.

But $\mu \cdot \omega^{\nu \cdot m} = \mu \cdot (\omega^{\nu})^m = \mu \cdot \omega^{\nu} \cdots \omega^{\nu}$.

Takeuti then inductively defines $\omega_0 = 1$ and $\omega_{n+1} = \omega^{\omega_n}$.

He proves that ω_k is $(n-k)$-accessible for $n > k$, by quantifier-free induction on k.

As a special case, we have that ω_k is accessible for every k.

It then follows that ϵ_0 is accessible.

It is crucial that each of these steps can be shown by a finitistically acceptable argument.

That is, they must be "effectively executed in principle. . . within the scope of a concrete treatment".

We are meant to see this by Gedankenexperimenten.

Mach, "On Thought Experiments" (1905)

[Those] whose ideas are good representations of the facts, will keep fairly close to reality in their thinking. Indeed, it is the more or less non-arbitrary representation of facts in our ideas that makes thought experiments possible.

But are these steps really thinkable in an effective, concrete way?

Gödel, "On an extension of finitary mathematics which has not yet been used" (1972)

The situation may be roughly described as follows: Recursion for ϵ_0 could be proved finitarily if the consistency of number theory could. On the other hand the validity of this recursion can certainly not be made *immediately* evident, as is possible, for example in the case of ω^2. That is to say, one cannot grasp at one glance the various structural possibilities which exist for decreasing sequences, and there exists, therefore, no *immediate* concrete knowledge of the termination of every such sequence. But furthermore such concrete knowledge (in Hilbert's sense) cannot be realized either by a stepwise transition from smaller to larger ordinal numbers, because the concretely evident steps, such as $\alpha \to \alpha^2$, are so small that they would have to be repeated ϵ_0 times in order to reach ϵ_0.

Takeuti, "Consistency Proofs and Ordinals" (1975)

This proof is very clear and transparent if one is familiar with the primitive recursive structure of the ordinals less than ϵ_0.

How to resolve this conflict?

Gödel's problem with Takeuti's argument is phenomenological; it takes issue with the surveyability capacities that he thinks the argument supposes.

But Takeuti's argument works on ordinal notations, not ordinals themselves.

Its reliance on ordinal notations, though, raises another problem: the system of ordinal notations used here, Kleene's O, is known to be a Π^1_1 set.

Can we finitistically prove things about terminating decreasing sequences in O in light of this set's complexity?

Takeuti, "Axioms of Arithmetic and Consistency" (Sugaku Seminar, 1994)
There is not much reason to oppose this idea by claiming that the notion that all decreasing sequences terminate within finite steps is a Π^1_1 notion in Kleene's hierarchy. What is important is not which hierarchy the notion belongs to, but how clear it is.

I want to develop Takeuti's defense against the complexity objection in two ways, from a historical point of view, and a computational point of view.

Descartes, *La géométrie* (1637)

It seems very clear to me that if we make the usual assumption that geometry is precise and exact, while mechanics is not; and if we think of geometry as the science which furnishes a general knowledge of the measurement of all bodies, then we have no more right to exclude the more complex curves than the simpler ones, provided they can be conceived of as described by a continuous motion or by several successive motions, each motion being completely determined by those which precede; for in this way an exact knowledge of the magnitude of each is always obtainable.

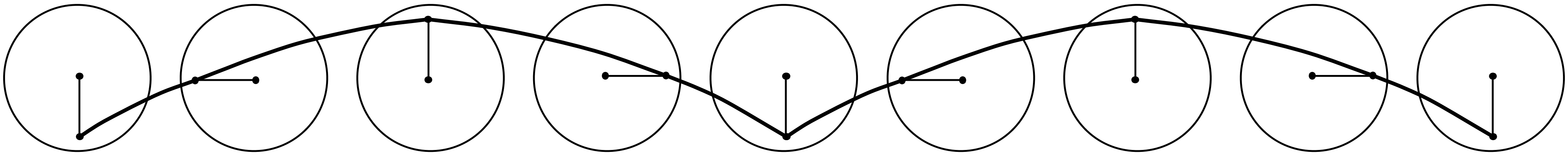
The curves excluded, called "mechanical" by Descartes, are today called transcendental.

Mechanical curves include the spiral, and generally curves defined by logarithms, exponentials, and trigonometry, as well as other "special functions".

La Géométrie (1637)
All the points of the curves we call geometric, that's to say have a precise and exact measure, necessarily have some relation to the points of a straight line that can be expressed by some equation.

These two criteria of geometricity, the kinematic and algebraic criteria, are extensionally equivalent: they pick out the same classes of curves.

Descartes maintains that curves geometrical in his sense(s) are better known than mechanical / transcendental curves.



$$x = r\cos^{-1}\left(1 - \frac{y}{r}\right) - \sqrt{y(2r - y)}$$

This curve, the cycloid, is a mechanical curve: it has no algebraic definition.

Leibniz, *De geometria recondita et analysi indivisibilium atque infinitorum*, 1686
Certainly it is necessary to accept into geometry those lines. . . that can be drawn exactly and by a continuous motion, as seems clear for the cycloid and others like it,; therefore these lines should be considered not mechanical, but geometrical, especially since in their usefulness they leave the lines of common geometry (except for the circle and straight line) many leagues behind and they have properties of great importance that are directly capable of geometrical demonstration.. . . Descartes was no less in error for excluding them from geometry than were the Ancients. . .

Newton independently gave a similar argument (1707).

Descartes identified a formal criterion (actually, two) for a curve being knowable exactly.

Leibniz and Newton argued that some curves not meeting that criterion are just as well known as those that do, using a criterion of clarity not captured by Descartes' criterion.

That is, they stepped around the alleged epistemic importance of Descartes' formal criterion and argued instead from epistemic basics themselves.

Takeuti: "What is important is not which hierarchy the notion belongs to, but how clear it is."

Takeuti seems to be doing something similar.

While Takeuti describes the complexity of the ordering as Π^1_1, Rathjen (2014) has noted that the particular usage of accessibility needed is only Π^0_2.

Kelly (1995) has noted that a Π^0_1 set A is refutable with certainty: to determine if $x \in \overline{A}$, ask if $(\exists y)R(x, y)$ for R computable. In the limit (that is, after finitely many errors), we will arrive at such a y if one exists simply by going through all possible values of y in $\mathbb{N}$.

Similarly, a Π^0_2 set A is refutable in the limit: to determine if $x \in \overline{A}$, ask if $(\exists y)(\forall z)R(x, y, z)$ for R computable. In the limit we will arrive at a y such that $(\forall z)R(x, y, z)$ if one exists, and $(\forall z)R(x, y, z)$ designates a relation refutable with certainty.

Thus we have a clearer grasp of the computational strength of accessibility: it is refutable in the limit.

That's to say that if there is a counterexample to an ordinal's being accessible, we will find it in a computable way in the limit.

Does such knowledge of the accessibility of an ordinal satisfy the epistemic demands of finitism—or, better, the "sharpened" type of finitism stressed by Takeuti, following Hilbert-Bernays?

We could characterize the epistemic position of Takeuti as surveyability: an ordinal is knowable as accessibility in a finitistically acceptable way if its accessibility can be surveyed by an agent like us, where surveying may involve a process only completable in the limit (hence, "ability").

A downside of this position is that it depend on the particular strengths of an individual knower to survey such a collection, and this may not be shared by all agents.

We may read Gödel's objection in this light.

If this is right, then this sharpening of finitism loses the quality stressed by Hilbert, that finitary reasoning is the core type of reasoning common to all scientific knowledge (and hence knowers).

Takeuti, "Consistency Proofs and Ordinals" (1975)
Although the accessibility proof described above is clear its formalization is not simple. The same method is repeated many, many times and the number of repetitions becomes higher and higher as the ordinal becomes larger and larger.

A next step of the project is to examine formalizations of Takeuti's argument to see what other insights about its epistemic value can be mined.