Eta-equalities in Martin-Löf type theory

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Martin-Löf type theory I

Forms of judgement

$$\alpha : \textbf{type}$$
$$\alpha = \beta : \textbf{type}$$
$$a : \alpha$$
$$a = b : \alpha$$

Rules of type formation

$$\textbf{set} : \textbf{type} \qquad \frac{A : \textbf{set}}{A : \textbf{type}} \qquad \frac{\alpha : \textbf{type} \qquad x : \alpha \vdash \beta : \textbf{type}}{(x : \alpha)\beta : \textbf{type}}$$

Martin-Löf type theory II

Function application

$$\frac{f : (x : \alpha)\beta \qquad a : \alpha}{f(a) : \beta[a/x]} \qquad \frac{f = g : (x : \alpha)\beta \qquad a = a' : \alpha}{f(a) = g(a') : \beta[a/x]}$$

Abstraction

$$\frac{x : \alpha \vdash b : \beta}{[x]b : (x : \alpha)\beta} \qquad \frac{x : \alpha \vdash b = b' : \beta}{[x]b = [x]b' : (x : \alpha)\beta}$$

Eta

$$[x]f(x) = f : (x : \alpha)\beta$$

Martin-Löf type theory III

Π-formation

$$\Pi : (X : \mathbf{set})((X)\mathbf{set})\mathbf{set}$$

Π-introduction

$$\lambda : (X : \mathbf{set})(Y : (X)\mathbf{set})((x : X)Y(x))\Pi(X, Y)$$

Π-elimination

$$\frac{c : \Pi(A, B) \qquad C : (\Pi(A, B))\mathbf{set} \qquad d : (z : (x : A)B(x))C(\lambda(z))}{\mathbf{F}(A, B, C, c, d) : C(c)}$$

Π-equality

$$\mathbf{F}(A, B, C, \lambda(A, B, f), d) = d(f) : C(\lambda(A, B, f))$$

Eta-equalities

- Lambda calculus:

$$\lambda x.fx = f$$

- Martin-Löf type theory, lower types (**set**s):

$$\lambda([x]\mathbf{ap}(c,x)) = c : \Pi(A,B)$$

$$\langle \mathbf{fst}c, \mathbf{snd}c\rangle = c : \Sigma(A,B)$$

In general, let the *n*-ary function **con** be the unique constructor of A. Assume that we have n "inverse functions"

$$\mathbf{invcon}_k(\mathbf{con}(a_1,\ldots,a_n)) = a_k : \alpha$$

The η-equality for A is then

$$\mathbf{con}(\mathbf{invcon}_1(c),\ldots,\mathbf{invcon}_n(c)) = c : A$$

- Martin-Löf type theory, higher types:

$$[x]f(x) = f : (x:\alpha)\beta$$

A puzzling situation

- Lower-order eta-equalities are justified by Martin-Löf's meaning explanations.
- Yet they are not part of the canonical version (1986) of Martin-Löf type theory.
- The higher-order eta-equality is a part of canonical MLTT.

This raises two questions

1. Why not accept a principle that is justified by the meaning explanations?
2. What justifies the stipulation of higher-order eta?

Definitional equality

In canonical MLTT judgemental equality

$$\alpha = \beta : \textbf{type} \qquad a = b : \alpha$$

is to be understood as definitional equality.

This understanding may not be forced upon one by the meaning explanations.

But it is motivated by the conception of types and objects as meaningful expressions.

The principles of individuation of types and objects are determined by the rules governing judgemental equality.

Definitional equality is equality of meaning, synonymy.

The principles of definitional equality

Curry & Feys and Martin-Löf characterize definitional equality as the equivalence relation $\equiv$ on terms generated by axioms

$$\textit{definiendum} \equiv \textit{definiens}$$

and the rule

$$\frac{X \equiv Y \qquad Z \equiv Z'}{X \equiv Y'}$$

where Y' is the result of replacing an occurrence of Z in Y by Z'.

When the language includes variable-binding operations we may add renaming of bound variables.

Soundness for definitional equality

The understanding of judgemental equality as definitional equality requires that all rules and axioms be sound for definitional equality.

Judgements $\alpha = \beta : \textbf{type}$ and $a = b : \alpha$ are sound for definitional equality if the following inferences are justified.

$$\frac{\alpha = \beta : \textbf{type}}{\alpha \equiv \beta} \qquad \frac{a = b : \alpha}{a \equiv b}$$

A rule

$$\frac{J_1 \quad \ldots \quad J_n}{a = b : \alpha}$$

is sound for definitional equality if from the assumption that all premisses J_k are sound we may infer that the conclusion $a = b : \alpha$ is sound.

Eta and definitional equality

Claim

Eta-equalities at lower order are not sound for definitional equality.

Consider

$$\langle \mathbf{fst} c, \mathbf{snd} c \rangle = c : \Sigma(A, B)$$

For this judgement to be sound for definitional equality, it would have to be of the form

$$\textit{definiendum} \equiv \textit{definiens}$$

But $\langle -, - \rangle$ is a constructor and admits of no definition.

And the projection functions **fst** and **snd** are already defined.

Argument removal

The following rule is admissible:

$$\frac{x : \alpha \vdash f(x) : \beta}{f : (x : \alpha)\beta}$$

At the level of terms this is an instance of argument removal:

$$f(a) \dashrightarrow f$$

Compare Frege's formation rule in *Grundgesetze* § 26:

> From a name a, with b as part, one obtains a function name by leaving out b at one or more places of its occurrence.

In mathematical practice it is common to regard the equation

$$f(x) \equiv t[x]$$

as defining the function f.

Definitional equality reconsidered

The characterization of definitionally equality must be sensitive to the underlying language.

When argument removal is present it may be that novel principles of definitional equality must be recognized.

Indeed, if

$$f(x) \equiv x^2 + x + 1$$

is to be regarded as defining f, then it seems that higher-order eta must be assumed.

$$\dfrac{\dfrac{[x]f(x) \equiv f}{f \equiv [x]f(x)} \qquad \dfrac{f(x) \equiv x^2 + x - 1}{[x]f(x) \equiv [x](x^2 + x - 1)}}{f \equiv [x](x^2 + x - 1)}$$

Conclusions

- The stipulation of lower-order eta-equalities as axioms does not threaten the simple-minded consistency of the system.
- But such stipulation is not sound for definitional equality.
- With argument removal present higher-order eta is a primitive principle of definitional equality.
- Hence, higher-order eta is sound for definitional equality.