# Ishihara's Contributions to Constructive Analysis

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## The framework

*Bishop-style constructive mathematics* (**BISH**):

*mathematics with intuitionistic logic*

and some appropriate set- or type-theoretic foundation such as

– the **CST** of Myhill, Aczel, and Rathjen;

– the Constructive Morse Set Theory of Bridges & Alps;

– Martin-Löf type theory.

We also accept **dependent choice**,

*If $S$ is a subset of $A \times A$, and for each $x \in A$ there exists $y \in A$ such that $(x, y) \in S$, then for each $a \in A$ there exists a sequence $(a_n)_{n \geqslant 1}$ such that $a_1 = a$ and $(a_n, a_{n+1}) \in S$ for each $n$,* We also accept **dependent choice**,

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and hence **countable choice,**

*If $X$ is an inhabited set, $S$ is a subset of $\mathbf{N}^+ \times X$, and for each positive integer $n$ there exists $x \in X$ such that $(n, x) \in S$, then there is a function $f : \mathbf{N}^+ \to X$ such that $(n, f(n)) \in S$ for each $n \in \mathbf{N}^+$.*

## The aim

To present some of Ishihara's fundamental contributions to Bishop-style constructive analysis, and their consequences.

# Part I

# Ishihara's Tricks and BD-N

# Our first results together

A linear mapping $T : X \to Y$ between normed spaces is **well behaved** if for each $x \in X$,

$$\forall y \in \ker T (x \neq y) \Rightarrow Tx \neq 0.$$

where $a \neq 0$ means $\|a\| > 0$.

**Fact:** If every bounded linear mapping between normed spaces is well behaved, then we can prove **Markov's Principle (MP)** in the form

$$\forall x \in \mathbf{R}(\neg(x = 0) \rightarrow |x| > 0).$$

To see this, consider $T : x \rightsquigarrow ax$ on $\mathbf{R}$, where $\neg(a = 0)$: $\ker T = \{0\}$, $1 \neq 0$, and $T1 = a$.

**Theorem 1** *A linear mapping $T$ of a normed space $X$* onto *a Banach space $Y$ is well behaved.*

**Sketch proof.** Consider $x \in X$ such that $x \neq y$ for each $y \in \ker T$. Construct a binary sequence $(\lambda_n)$ such that

$$\begin{aligned} \lambda_n &= 1 \rightarrow \|Tx\| < 1/n^2, \\ \lambda_n &= 0 \Rightarrow \|Tx\| > 1/(n+1)^2. \end{aligned}$$

WLOG $\lambda_1 = 1$.

If $\lambda_n = 1$, set $t_n = 1/n$; if $\lambda_{n+1} = 1 - \lambda_n$, set $t_k = 1/n$ for all $k \geq n$.

Then $(t_n)$ is a Cauchy sequence and therefore has a limit $t$ in $\mathbf{R}$.

OTOH, $\sum \lambda_n Tx$ converges to a sum $z$ in $Y$, by comparison with $\sum 1/n^2$. Let $y = x - tz$.

Show that $Ty = 0$ (details omitted). Then

$$tz = x - y \neq 0,$$

so $t > 0$ and $\|z\| > 0$.

Pick $N$ such that for all $n \geq N$, $t_n > N^{-1}$ and therefore $t_n \|z\| > N^{-1} \|z\|$.

If $\lambda_{N+1} = 1$, then

$$t_{N+1} \|z\| = (N+1)^{-1} \|z\| < N^{-1} \|z\|$$

—absurd. Thus $\lambda_{N+1} = 0$ and $\|Tx\| > 1/(N+1)^2$. $\square$

A subset $S$ of a metric space $(X, \rho)$ is **located** if

$$\rho(x, S) = \mathsf{inf}\{\rho(x, y) : y \in S\}$$

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**Theorem 2** *Let $T$ be a linear mapping of a Banach space $X$ into a normed space $Y$. Let $B$ be a subset of* graph$(T)$ *that is closed and located in $X \times Y$, and let $(x, y) \in X \times Y$ be such that $y \neq Tx$. Then $\rho((x, y), B) > 0$.*

A mapping $f : X \to Y$ between metric spaces is **strongly extensional** if $f(x) \neq f(x')$—that is, $\rho(f(x), f(x')) > 0$—implies that $x \neq x'$.

**Corollary** *A linear mapping of a Banach space into a normed space is strongly extensional.*

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**Corollary** *A linear mapping of a Banach space into a normed space is strongly extensional.*

Note: For a linear mapping $T$, strong extensionality is equivalent to

$$(Tx \neq 0 \Rightarrow \forall z \in \ker T (x \neq z)).$$

So the Corollary is a kind of dual to Theorem 1.

## Ishihara's Tricks

*Continuity and Nondiscontinuity in Constructive Mathematics*, JSL **56**(4), 1991.

**Ishihara's first trick** *Let $f$ be a strongly extensional mapping of a complete metric space $X$ into a metric space $Y$, and let $(x_n)$ be a sequence in $X$ converging to a limit $x$. Then for all positive $a, b$ with $a < b$, either $\rho(f(x_n), f(x)) > a$ for some $n$, or else $\rho(f(x_n), f(x)) < b$ for all $n$.*

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**Ishihara's second trick:** *Let $f$ be a strongly extensional mapping of a complete metric space $X$ into a metric space $Y$ and let $(x_n)$ be a sequence in $X$ converging to a limit $x$. Then for all positive $a, b$ with $a < b$, either $\rho(f(x_n), f(x)) > a$ for infinitely many $n$, or else $\rho(f(x_n), f(x)) < b$ for all sufficiently large $n$.*

A mapping $f : X \to Y$ between metric spaces is

- **sequentially continuous at** $x \in X$ if $x_n \to x$ implies that $f(x_n) \to f(x)$;

- **sequentially nondiscontinuous at** $x \in X$ if $x_n \to x$ and $\rho(f(x_n), f(x)) \geq \delta$ for all $n$ together imply that $\delta \leq 0$.

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Sequentially continuous, and sequentially nondiscontinuous, *on* $X$ have the obvious meanings.

**Theorem 3** *A mapping of a complete metric space $X$ into a metric space $Y$ is sequentially continuous if and only if it is both sequentially nondiscontinuous and strongly extensional.*

A real number $a$ is said to be **pseudopositive** if

$$\forall yx \in \mathbf{R}(\neg\neg(0 < x) \vee \neg\neg(x < a)).$$

The **Weak Markov Principle (WMP)** states that every pseudopositive real number is positive, and is a consequence of **MP**.

**Theorem 4** *The following are equivalent.*

1. *Every mapping of a complete metric space into a metric space is strongly extensional.*

2. *Every sequentially nondiscontinuous mapping of a complete metric space into a metric space is sequentially continuous.*

3. **WMP**.

It is now simple to prove a form of **Kreisel-Lacombe-Schoenfield-Tseitin Theorem:**

**Theorem 5** *Under the Church-Markov-Turing Thesis, the following are equivalent:*

1. *Every mapping of a complete metric space into a metric space is sequentially continuous.*

2. **WMP.**

The original KLST theorem deletes ‘sequentially’ from (1) and ‘W’ from (2).

Recall the essentially nonconstructive **limited principle of omniscience (LPO):**

$$\forall \mathbf{a} \in 2^{\mathbf{N}}(\forall n(a_n = 0) \vee \exists n(a_n = 1))$$

**Ishihara's third trick:** *Let $f$ be a strongly extensional mapping of a complete metric space $X$ into a metric space $Y$, let $(x_n)$ be a sequence in $X$ converging to a limit $x$, and let $a > 0$. Then*

$$\forall n \exists k \geq n(\rho(f(x_n), f(x)) > a) \Rightarrow \textbf{ LPO}).$$

This trick was introduced in

*A constructive version of Banach's inverse mapping theorem*, NZJM **23**, 71–75, 1994.

Consider the following not uncommon situation.

Given a strongly extensional mapping $f$ of a complete metric space $X$ into a metric space $Y$, a sequence $(x_n)$ in $X$ converging to a limit $x$, and a positive $\varepsilon$, we want to prove that $\rho(f(x_n), f(x)) < \varepsilon$ for all sufficiently large $n$.

According to Ishihara's second trick, either we have the desired conclusion, or else $\rho(f(x_n), f(x)) > \varepsilon/2$ for all sufficiently large $n$.

In the latter event, according to Ishihara's third trick, we can derive **LPO**.

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*In many instances, we can prove that*

$$\textbf{LPO} \Rightarrow \neg\forall n \exists k \geq n(\rho(f(x_n), f(x)) > a),$$

*thereby ruling out the undesired alternative in Ishihara's second trick.*

The first example of this trick was in Ishihara's proof of the constructive Banach inverse mapping theorem:

**Theorem 6** *Let $T$ be a one-one, sequentially continuous linear mapping of a separable Banach space onto a Banach space. Then $T^{-1}$ is sequentially continuous.*

We shall discuss shortly another remarkable insight of Ishihara's, which will explain why we cannot delete 'sequentially' from the conclusion of Theorem 6 even when we delete it from the premisses. Before doing so, we remark Hannes Diener's interesting extension of Ishihara's tricks.

Let $X$ be a metric space. For each sequence $x \equiv (x_n)$ in $X$ converging to $x_\infty \in X$, and each increasing binary sequence $\lambda \equiv (\lambda_n)$, Diener defines a sequence $\lambda \circledast x$ by

$$(\lambda \circledast x)_n = \begin{cases} x_m & \text{if } \lambda_n = 1 \text{ and } \lambda_m = 1 - \lambda_{m-1} \\ \\ x_\infty & \text{if } \lambda_n = 0. \end{cases}$$

Then $\lambda \circledast x$ is a Cauchy sequence. We say that $X$ is **complete enough** if for every such $x, x_\infty$, and $\lambda$, the sequence $\lambda \circledast x$ converges in $X$.

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**Fact 4:** Ishihara's three tricks hold with 'complete' replaced by 'complete enough'.

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**Fact 4:** Ishihara's three ricks hold with 'complete' replaced by 'complete enough'.

**Application:** A proof, under a special extra-Bishop assumption, of the **Riemann permutation theorem**: *if every rearrangement of a series of real numbers converges, then the series is absolutely convergent.*

It is to that extra-Bishop condition that we now turn.

## Pseudoboundedness and BD-$\mathbf{N}$

Another seminal paper of Ishihara's:

> *Continuity properties in constructive mathematics*, JSL **57**(2), 1992, 557–565.

A subset $A$ of $\mathbf{N}$ is **pseudobounded** if $\lim_{n\to\infty} \dfrac{a_n}{n} = 0$ for every sequence $(a_n)$ in $A$.

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A principle of countable boundedness,

> **BD**-$\mathbf{N}$ : *Every inhabited, countable, pseudobounded subset of* $\mathbf{N}$ *is bounded.*

**BD**-$\mathbb{N}$ is derivable using the law of excluded middle. Ishihara showed that **BD**-$\mathbb{N}$ is derivable under the Church-Markov-Turing thesis and **MP**. It is also derivable using Brouwer's continuity principles, and so holds intuitionistically. But, as shown first by Lietz,

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Thus a theorem of the type

$$\textbf{BISH} \vdash (P \Rightarrow \textbf{BD-}\mathbf{N})$$

proves the impossibility of ever finding a proof of

$$\textbf{BISH} \vdash P$$

Ishihara's links between pseudoboundedness and sequential continuity.

**Ishihara's link 1** *Let $A$ be an inhabited, pseudobounded subset of $\mathbf{N}$. Then there exist a complete subset $X$ of $\mathbf{R}$ and a sequentially continuous mapping $f : X \to \{0, 1\}$ such that*

$$0 \in X \wedge f(0) = 0 \wedge \forall m(m \in A \to 2^{-m} \in X \wedge f(2^{-m}) = 1).$$

*If also $A$ is countable, then $X$ is separable.*

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*If also $A$ is countable, then $X$ is separable.*

**Ishihara's link 2** *Let $f$ be a sequentially continuous mapping of a metric space $X$ into a metric space $Y$. Then for each $x \in X$ and $\varepsilon > 0$, there exists an inhabited, pseudobounded subset $A$ of $\mathbf{N}$ such that*

$$\forall m > 0(\exists x' \in X(\rho(x, x') < m^{-1} \wedge \rho(f(x), f(x')) > \varepsilon) \Rightarrow m \in A).$$

*If also $X$ is separable, then $A$ is countable.*

**Theorem 8.** *The following are equivalent (over* **BISH***).*

**(i)** *Every sequentially continuous mapping of a separable metric space into a metric space is continuous.*

**(ii)** *Every sequentially continuous mapping of a complete, separable metric space into a metric space is continuous.*

**(iii)** **BD**-$\mathbb{N}$.

**Proof.**

**(i)** $\Rightarrow$ **(ii)**: Trivial.

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**(ii)** $\Rightarrow$ **(iii).** Let $A \subset \mathbf{N}$ be inhabited, countable, pseudobounded. By Ishihara's Link 1, there exist a complete, separable $X \subset \mathbf{R}$ and a sequentially continuous $f : X \to \{0, 1\}$ such that

$$0 \in X \wedge f(0) = 0 \wedge \forall m(m \in A \to 2^{-m} \in X \wedge f(2^{-m}) = 1).$$

Assuming (ii), we can find $N \in \mathbf{N}$ such that if $x \in X$ and $|x| < 2^{-N}$, then $|f(x)| < 1$.

If $m \in A$ and $m \geq N$, then $2^{-m} \in X$ and $\left|2^{-m}\right| < 2^{-N}$, so $1 = f(2^{-m}) < 1$, a contradiction. Hence $m \leq N$ for all $m \in A$.

**Proof.**

**(i)** $\Rightarrow$ **(ii)**: Trivial.

**(ii)** $\Rightarrow$ **(iii).** Let $A \subset \mathbf{N}$ be inhabited, countable, pseudobounded. By Ishihara's Link 1, there exist a complete, separable $X \subset \mathbf{R}$ and a sequentially continuous $f : X \longrightarrow \{0, 1\}$ such that

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**(iii)** $\Rightarrow$ **(i).** Use Ishihara's Link 2. $\square$

A few examples of statements equivalent to **BD–$\mathbf{N}$** over **BISH** (and therefore derivable classically, intuitionistically, and recursively).

1. *Every one-one bounded linear mapping of a separable Banach space onto a Banach space has continuous (bounded) inverse.*

2. *Every sequence of bounded linear mappings from a separable Banach space into a normed space is equicontinuous.*

3. *The locally convex space $\mathcal{D}(\mathbf{R})$ of all infinitely differentiable functions $f : \mathbf{R} \to \mathbf{R}$ with compact support is sequentially complete.*

A mapping $f : X \longrightarrow Y$ between metric spaces is **uniformly sequentially continuous** if for any sequences $(x_n), (x'_n)$ in $X$,

$$\rho(x_n, x'_n) \to 0 \Rightarrow \rho(f(x_n), f(x'_n)) \to 0.$$

A mapping $f : X \longrightarrow Y$ between metric spaces is **uniformly sequentially continuous** if for any sequences $(x_n), (x'_n)$ in $X$,

$$\rho(x_n, x'_n) \to 0 \Rightarrow \rho(f(x_n), f(x'_n)) \to 0.$$

4. *Every uniformly sequentially continuous mapping of a complete separable metric space into a metric space is uniformly continuous.*

5. *Every uniformly sequentially continuous mapping of a complete separable metric space into a metric space is pointwise continuous.*

6. *If* $T$ *is a nonzero bounded linear mapping of a separable Hilbert space into itself such that* $T^*$ *exists and* range$(T)$ *is complete, then* $T$ *is an open mapping.*

7. *Every one-one, selfadjoint, sequentially continuous linear mapping from a Hilbert space onto itself is bounded.*

# Part II

# Ishihara and Functional Analysis

# Hahn-Banach and separation theorems

A linear functional $u$ on a normed space $X$ is **normable**, or **normed**, if

$$\|u\| \equiv \mathsf{sup}\,\{\|u(x) : x \in X, \|x\| \leq 1\|\}$$

exists.

A nonzero bounded linear functional on a normed space is normable if and only if its kernel is located (Bishop).

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Bishop's Hahn-Banach theorem:

> *Let $v$ be a nonzero bounded linear functional on a linear subset $Y$ of a separable normed space $X$ such that $\ker v$ is located in $X$. Then for each $\varepsilon > 0$ there exists a normable linear functional $u$ on $X$ such that $u(y) = v(y)$ for all $y \in Y$ and $\|u\| < \|v\| + \varepsilon$.*

H. Ishihara, *On the constructive Hahn-Banach theorem*, Bull. London. Math. Soc. **21**, 79–81, 1989

Let $X$ be a normed space.

The norm on a $X$ is **Gâteaux differentiable** if

$$\lim_{t \to 0} \frac{\|x + ty\| - \|y\|}{t}$$

exists for all unit vectors $x, y$ in $X$.

We say that $X$ is **uniformly convex** if for each $\varepsilon > 0$ there exists $\delta > 0$ such that for all unit vectors $x, y \in X$, $\left\| \frac{1}{2}(x + y) \right\| \leq 1 - \delta$ whenever $\|x - y\| > \varepsilon$.

**Theorem 7** *Let $X$ be a uniformly convex Banach space with Gâteaux differentiable norm, and let $v$ be a nonzero normable linear functional on a linear subspace $Y$ of $X$. Then there exists a unique normable linear functional $u$ on $X$ such that $u(y) = v(y)$ for all $y \in Y$ and $\|u\| = \|v\|$.*

In the context of a uniformly convex Banach space with Gâteaux differentiable norm, Ishihara also removed an '$\varepsilon$' from the conclusion of Bishop's separation theorem.

These results apply, in particular, to Hilbert spaces.

H. Ishihara, *Locating subsets of a Hilbert space*, Proc. AMS **129**(5), 1385–1390, 2000.

**Theorem 8** *Let $C$ be an inhabited, bounded convex subset of an inner product space $X$. Then $C$ is located if and only if*

$$\mathsf{sup}\,\{\mathsf{Re}\,\langle x, v\rangle : v \in X\}$$

*exists for each $x \in X$.*

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Remarkably, when $X$ is a Hilbert space, the 'bounded' hypothesis is unnecessary. Ishihara's proof of this uses a very ingenious '$\lambda$' argument.

In consequence, he proves

**Theorem 9** *If $T$ is an operator with an adjoint on a Hilbert space, then the image under $T$ of the unit ball is located.*

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Taken with a prior result of Richman, this leads to

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Note: The proposition every bounded operator on a Hilbert space has an adjoint' implies **LPO**.

## Smoothness, duality, locatedness

A normed space $X$ is

- **smooth** if its norm is Gâteaux differentiable at each nonzero vector;
- **uniformly smooth** if it is smooth and for each $\varepsilon > 0$ there exists $\delta > 0$ such that
$$\left| u_x(y) - \frac{\|x + ty\| - \|y\|}{t} \right| < \varepsilon$$
whenever $x, y$ are unit vectors in $X$ and $0 < |t| < \delta$.

Inner product spaces are uniformly smooth.

In *Locating subsets of a normed space* (H. Ishihara and L.S. Vîță, Proc. AMS **131**(10), 3231–3239, 2003) the authors extend and generalise much of Ishihara's earlier work on locatedness in Hilbert spaces.

Considerable technical complexities lead to and through the proof of the following results in that paper.

**Theorem 11** *A separable normed space is uniformly smooth if and only if it has a uniformly convex dual.*

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**Theorem 11** *A separable normed space is uniformly smooth if and only if it has a uniformly convex dual.*

**Theorem 12** *Let $X$ be a uniformly convex, uniformly smooth Banach space over $\mathbf{R}$, and let $C$ be an inhabited, bounded, convex subset of $X$. Then $C$ is located if and only if*

$$\sup\{f(y) : y \in C\}$$

*exists for each normable linear functional $f$ on $X$.*

## Our last joint paper

D.S. Bridges, H. Ishihara, and M. McKubre-Jordens, *Uniformly convex Banach spaces are reflexive—constructively*, MLQ **59**(4–5), 352–356, 2013.

Let $X$ be a normed space, and $X^*$ the linear space of all bounded linear functionals on $X$. In the infinite-dimensional case, the normability of every element of $X^*$ implies **LPO**, so we cannot describe $X^*$ as a normed space in the familiar classsical way.

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It is, however, an example of a *quasinormed space*, in which there is a notion of *normability* that corresponds to the existence of the usual classical norm. In turn, the linear space $X^{**}$ of all bounded linear functionals on $X^*$ is a quasinormed space.

Classically, the notion of a quasinorm is essentially equivalent to that of a norm.

A normed space $X$ is **reflexive** if for each normable element $F$ of its second dual $X^{**}$, there exists a (perforce unique) $x$ in $X$ such that $F = \widehat{x}$, where

$$\widehat{x}(u) \equiv u(x) \quad (u \in X^*).$$

The classical **Milman-Pettis theorem** says that:

*A uniformly convex Banach space is reflexive, and the mapping $x \rightsquigarrow \widehat{x}$ is a norm-preserving bijection.*

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Our general constructive counterpart applies to complete, pliant, uniformly convex quasinormed spaces.

Pliancy is a constructive condition that holds trivially for all normed spaces in classical mathematics. Constructively, a separable normed space is pliant, as is a normed space with Gâteaux differentiable norm.

A particular corollary of our general theorem is:

**Theorem 14** *A uniformly convex Banach space is reflexive under either of these conditions:*

**(i)** *it is separable;*

**(ii)** *it has Gâteaux differentiable norm.*

In particular, a Hilbert space is reflexive; but that is essentially a consequence of the Riesz Representation Theorem.

## Conclusion

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**May there be many happy returns!**

dsb, Kanazawa meeting for Ishihara's 60th 010318