Second Workshop on Mathematical Logic and its Applications
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Hilbert's ϵ-operator in categorical logic

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Primary doctrines

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Example: contravariant powerset functor

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$\mathcal{P}(A)$ is ordered by $\subseteq$

Finite meets are $\cap$

Elementary and existential doctrines

[F.W. Lawvere]

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When P is $\mathcal{P}$ $\quad \delta_A = \{(a,b) \in A \times A \mid a = b\}$ *[F.W. Lawvere]*

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Comprehension schema: for $\alpha \in \mathcal{P}(A)$

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Abstract characterization in the framework of doctrines.

Completions

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Comprehension completion: the comprehension schema can be freely added to any doctrine.

$$P_c\colon \mathcal{C}_c^{op} \to \mathcal{I}nf\mathcal{SL}$$

[Grothendieck's construction of vertical morphisms.]

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[Grothendieck's construction of vertical morphisms.]

Elementary quotient completion: effective quotients can be freely added to any elementary existential doctrine.

$$\widehat{P}\colon \mathcal{Q}_P^{op} \to \mathcal{InfSL}$$

[M.E. Maietti and G. Rosolini. Elementary quotient completion. 2013]

Back to triposes

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Tripos

$\downarrow$

Topos

Back to triposes

Tripos $\quad P\colon \mathcal{C}^{op} \to \mathcal{InfSL}$

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Tripos $\downarrow$ **Topos**

$$\begin{array}{c} P\colon \mathcal{C}^{op} \to \mathit{InfSL} \\ \downarrow \\ P_c\colon \mathcal{C}_c^{op} \to \mathit{InfSL} \end{array}$$

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$$\widehat{P_c}\colon \mathcal{Q}_{\mathcal{C}_c}^{op} \to \mathit{InfSL}$$

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Topos

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$$\begin{array}{ccc}
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\Big\downarrow & & \Big\downarrow \\
 & & P_c\colon \mathcal{C}_c^{op} \to \mathcal{InfSL} \\
 & & \Big\downarrow \\
 & & \widehat{P_c}\colon \mathcal{Q}_{\mathcal{C}_c}^{op} \to \mathcal{InfSL} \\
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\end{array}$$

Theorem: $\mathcal{Q}_{P_c}$ is a topos iff $\widehat{P_c}$ satisfies the Rule of Unique Choice.

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Hilbert's ϵ-operator: P has Hilbert's ϵ-operator if for every $R \in P(A \times B)$ there is $\epsilon_R\colon A \to B$ such that

$$\exists_{\pi_A} R = P(\langle \mathrm{id}_A, \epsilon_R \rangle)(R)$$

Characterizations

$P\colon \mathcal{C}^{op} \to \mathcal{I}nf\mathcal{SL}$ is a tripos.

Theorem: $\hat{P}$ satisfies the Rule of Unique Choice if and only if P satisfies the Rule of choice

[Maietti & Rosolini. Relating quotient completions via categorical logic. 2016]

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Corollary: $\mathcal{Q}_{\hat{P}_c}$ is a topos if and only if P has Hilbert's ϵ-operator

Examples and future developments

$\mathcal{W}$ is a poset, $\perp \in \mathcal{W}$, $L = \mathcal{W}^{op}$ is a well order.

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$$\begin{array}{ccc} X & & L^X \\ {\scriptstyle f}\downarrow & & \uparrow{\scriptstyle \alpha \mapsto \alpha \circ f} \\ Y & & L^Y \end{array}$$

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The doctrine interprets **HA**.

Thank you

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