

Second Workshop on Mathematical Logic and its Applications
8 March 2018 - Kanazawa - Japan

Hilbert's ϵ -operator in categorical logic

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Primary doctrines

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Example: contravariant powerset functor

$$\mathcal{P}: \mathit{Set}^{\mathit{op}} \longrightarrow \mathit{InfSL}$$

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$\mathcal{P}(A)$ is ordered by \subseteq

Finite meets are \cap

Elementary and existential doctrines

[F.W. Lawvere]

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such that

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When P is \mathcal{P} $\delta_A = \{(a, b) \in A \times A \mid a = b\}$ [F.W. Lawvere]

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Abstract characterization in the framework of doctrines.

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[Grothendieck's construction of vertical morphisms.]

Elementary quotient completion: effective quotients can be freely added to any elementary existential doctrine.

$$\widehat{P}: \mathcal{Q}_P^{op} \rightarrow \mathit{InfSL}$$

[M.E. Maietti and G. Rosolini. Elementary quotient completion. 2013]

Back to triposes

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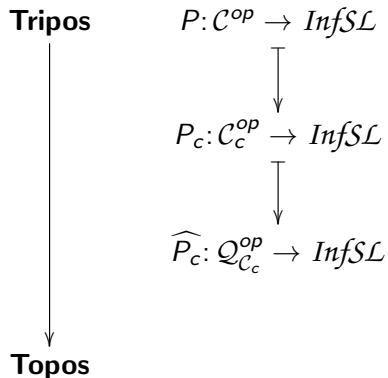


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$$\widehat{P}_c: \mathcal{Q}_{\mathcal{C}_c}^{op} \rightarrow \mathit{InfSL}$$

Back to triposes



Theorem: \mathcal{Q}_{P_c} is a topos iff \widehat{P}_c satisfies the Rule of Unique Choice.

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Hilbert's ϵ -operator: P has Hilbert's ϵ -operator if for every $R \in P(A \times B)$ there is $\epsilon_R: A \rightarrow B$ such that

$$\exists_{\pi_A} R = P(\langle \text{id}_A, \epsilon_R \rangle)(R)$$

Characterizations

$P: \mathcal{C}^{op} \rightarrow \text{InfSL}$ is a tripos.

Theorem: \hat{P} satisfies the Rule of Unique Choice if and only if P satisfies the Rule of choice

[Maietti & Rosolini. Relating quotient completions via categorical logic. 2016]

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[Maietti, Pasquali & Rosolini. Tripases, exact completions, and Hilbert's ϵ -operator. 2017]

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Theorem: P_c satisfies the Rule of Choice if and only if P has Hilbert's ϵ -operator

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Corollary: $\mathcal{Q}_{\hat{P}_c}$ is a topos if and only if P has Hilbert's ϵ -operator

Examples and future developments

\mathcal{W} is a poset, $\perp \in \mathcal{W}$, $L = \mathcal{W}^{op}$ is a well order.

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The doctrine interprets **HA**.

Thank you

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