Computational content of the fan theorem for coconvex bars

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Computational content of proofs

- Here: Proofs on sequences (i.e., of type $\mathbb{N} \to \iota$, $\mathrm{lev}(\iota) = 0$)

What is special for sequences $f : \mathbb{N} \to \iota$?

- Can be seen as streams, infinite type-0 objects.

Example: streams of booleans, $\mathbb{S}(\mathbb{B})$, with the single constructor

$$\mathrm{C} : \mathbb{B} \to \mathbb{S}(\mathbb{B}) \to \mathbb{S}(\mathbb{B})$$

Why consider streams?

- Reals naturally represented by streams of signed digits $-1, 0, 1$
- Supports access from the front ("most significant digit")
- Reduction of type levels

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Overview

- The model $\mathcal{C}$ of partial continuous functionals (Scott, Ershov)
- TCF (theory of computable functionals)
- Realizability, soundness theorem
- Computational content of the fan theorem for coconvex bars

Computable functionals

General view: computations are finite.

Arguments not only numbers and functions, but also functionals of any finite type.

- Principle of finite support. If $\mathcal{H}(\Phi)$ is defined with value n, then there is a finite approximation Φ_0 of Φ such that $\mathcal{H}(\Phi_0)$ is defined with value n.
- Monotonicity principle. If $\mathcal{H}(\Phi)$ is defined with value n and Φ' extends Φ, then also $\mathcal{H}(\Phi')$ is defined with value n.
- Effectivity principle. An object is computable iff its set of finite approximations is (primitive) recursively enumerable (or equivalently, Σ^0_1-definable).

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Information system $\mathbf{A} = (A, \mathrm{Con}, \vdash)$:

- A countable set of "tokens",
- Con set of finite subsets of A,
- $\vdash$ ("entails") subset of $\mathrm{Con} \times A$.

such that

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\begin{aligned}
&U \subseteq V \in \mathrm{Con} \to U \in \mathrm{Con},\\
&\{a\} \in \mathrm{Con},\\
&U \vdash a \to U \cup \{a\} \in \mathrm{Con},\\
&a \in U \in \mathrm{Con} \to U \vdash a,\\
&U, V \in \mathrm{Con} \to \forall_{a \in V}(U \vdash a) \to V \vdash b \to U \vdash b.
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$x \subseteq A$ is an ideal if

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&U \subseteq x \to U \in \mathrm{Con} \quad (x \text{ is consistent}),\\
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Function spaces

Let $\mathbf{A} = (A, \mathrm{Con}_A, \vdash_A)$ and $\mathbf{B} = (B, \mathrm{Con}_B, \vdash_B)$ be information systems. Define $\mathbf{A} \to \mathbf{B} := (C, \mathrm{Con}, \vdash)$ where

- $C := \mathrm{Con}_A \times B$,
- $\{ (U_i, b_i) \mid i \in I \} \in \mathrm{Con} :=$
  $\forall_{J \subseteq I}(\bigcup_{j \in J} U_j \in \mathrm{Con}_A \to \{ b_j \mid j \in J \} \in \mathrm{Con}_B)$
- $\{ (U_i, b_i) \mid i \in I \} \vdash (U, b)$ means $\{ b_i \mid U \vdash_A U_i \} \vdash_B b$.

$\mathbf{A} \to \mathbf{B}$ is an information system.

Application of an ideal r in $\mathbf{A} \to \mathbf{B}$ to an ideal x in $\mathbf{A}$ is defined by

$$\{ b \in B \mid \exists_{U \subseteq x} r(U, b) \}.$$

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(Free) algebras given by constructors:

$\mathbb{N}$	by	$0^{\mathbb{N}}, \mathrm{S}^{\mathbb{N} \to \mathbb{N}}$
$\alpha \times \beta$	by	$\langle ., . \rangle^{\alpha \to \beta \to \alpha \times \beta}$
$\alpha + \beta$	by	$(\mathrm{InL}_{\alpha\beta})^{\alpha \to \alpha + \beta}, (\mathrm{InR}_{\alpha\beta})^{\beta \to \alpha + \beta}$
$\mathbb{L}(\alpha)$	by	$\mathrm{Nil}^{\mathbb{L}(\alpha)}, \mathrm{Cons}^{\alpha \to \mathbb{L}(\alpha) \to \mathbb{L}(\alpha)}$
$\mathbb{S}(\alpha)$	by	$\mathrm{SCons}^{\alpha \to \mathbb{S}(\alpha) \to \mathbb{S}(\alpha)}$
$\mathbb{I}$	by	$\mathrm{Gen}^{\mathbb{I} \to \mathbb{I}}$

$\mathbb{S}(\alpha)$ and $\mathbb{I}$ have no nullary constructor, hence no "total" objects.

Information systems $\mathbf{C}_\rho = (C_\rho, \mathrm{Con}_\rho, \vdash_\rho)$

$\mathbf{C}_{\rho\to\sigma} := \mathbf{C}_\rho \to \mathbf{C}_\sigma$. At base types ι:

Tokens are type correct constructor expressions $\mathrm{C}a_1^* \dots a_n^*$. (Examples: 0, $C{*}0$, $C0{*}$, $C(C{*}0)0$.)

$U = \{a_1, \dots, a_n\}$ is consistent if

- all a_i start with the same constructor,
- (proper) tokens at j-th argument positions are consistent.

(Example: $\{C{*}0, C0{*}\}$.)

$U \vdash a$ (entails) if

- all $a_i \in U$ and also a start with the same constructor,
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(Example: $\{C{*}0, C0{*}\} \vdash C00$).

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Definition

- A partial continuous functional of type ρ is an ideal in $\mathbf{C}_\rho$.
- A partial continuous functional is computable if it is a (primitive) recursively enumerable set of tokens.

Ideals in $\mathbf{C}_\rho$: Scott-Ershov domain of type ρ.
Principles of finite support and monotonicity hold ("continuity").

- x^ι is total iff $x = \{\, a \mid \{b\} \vdash a \,\}$ for some token (i.e., constructor expression) b without $*$.
- x^ι is cototal iff every token $b(*) \in x$ has a "one-step extension" $b(\mathrm{C}\vec{*}) \in x$.

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- The model $\mathcal{C}$ of partial continuous functionals (Scott, Ershov)
- TCF (theory of computable functionals)
- Realizability, soundness theorem
- Computational content of the fan theorem for coconvex bars

A common extension T^+ of Gödel's T and Plotkin's PCF

Terms of T^+ are built from (typed) variables and (typed) constants (constructors C or defined constants D, see below) by (type-correct) application and abstraction:

$$M, N ::= x^\rho \mid \mathrm{C}^\rho \mid D^\rho \mid (\lambda_{x^\rho} M^\sigma)^{\rho\to\sigma} \mid (M^{\rho\to\sigma} N^\rho)^\sigma.$$

Every defined constant D comes with a system of computation rules, consisting of finitely many equations

$$D\vec{P}_i(\vec{y}_i) = M_i \qquad (i = 1, \dots, n)$$

with free variables of $\vec{P}_i(\vec{y}_i)$ and M_i among $\vec{y}_i$, where the arguments on the left hand side must be "constructor patterns", i.e., lists of applicative terms built from constructors and distinct variables.

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with free variables of $\vec{P}_i(\vec{y}_i)$ and M_i among $\vec{y}_i$, where the arguments on the left hand side must be "constructor patterns", i.e., lists of applicative terms built from constructors and distinct variables.

Examples

Fibonacci numbers

$$\mathrm{fib}(0) = 0,$$
$$\mathrm{fib}(1) = 1,$$
$$\mathrm{fib}(n+2) = \mathrm{fib}(n) + \mathrm{fib}(n+1).$$

The corecursion operator ${}^{\mathrm{co}}\mathcal{R}^{\tau}_{\mathbb{S}(\rho)}$ of type

$$\tau \to (\tau \to \rho \times (\mathbb{S}(\rho) + \tau)) \to \mathbb{S}(\rho)$$

is defined by

$${}^{\mathrm{co}}\mathcal{R}xf = \begin{cases} yz & \text{if } f(x) = \langle y, \mathrm{InL}(z)\rangle, \\ y({}^{\mathrm{co}}\mathcal{R}x'f) & \text{if } f(x) = \langle y, \mathrm{InR}(x')\rangle. \end{cases}$$

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Predicates and formulas

$$P, Q ::= X \mid \{\, \vec{x} \mid A \,\} \mid \mu_X(\forall_{\vec{x}_i}((A_{i\nu})_{\nu<n_i} \to X\vec{r}_i))_{i<k} \mid \nu_X(\dots)$$
$$A, B ::= P\vec{r} \mid A \to B \mid \forall_x A$$

Example: $\mathrm{Even} := \mu_X(X0, \forall_n(Xn \to X(\mathrm{S}(\mathrm{S}n))))$.

(Co)inductive predicates can be computationally relevant (c.r.) or non-computational (n.c). Example: $T_{\mathbb{N}}$ (c.r.) and $T_{\mathbb{N}}^{\mathrm{nc}}$ (n.c.)

Clauses and least-fixed-point (induction) axiom for $T_{\mathbb{N}}$:

$(T_{\mathbb{N}}^{+})_0 \colon 0 \in T_{\mathbb{N}}$
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$$T_{\mathbb{N}}^{-} \colon 0 \in X \to \forall_n(n \in T_{\mathbb{N}} \to n \in X \to \mathrm{S}n \in X) \to T_{\mathbb{N}} \subseteq X$$

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Coinductive predicates: ${}^{\mathrm{co}}T_{\mathbb{N}}$ (c.r.) and ${}^{\mathrm{co}}T_{\mathbb{N}}^{\mathrm{nc}}$ (n.c.)

Closure and greatest-fixed-point (coinduction) axioms for ${}^{\mathrm{co}}T_{\mathbb{N}}$:

$${}^{\mathrm{co}}T_{\mathbb{N}}^{-} : \forall_n (n \in {}^{\mathrm{co}}T_{\mathbb{N}} \to n \equiv 0 \vee \exists_{n'} (n' \in {}^{\mathrm{co}}T_{\mathbb{N}} \wedge n \equiv \mathrm{S}n'))$$

$${}^{\mathrm{co}}T_{\mathbb{N}}^{+} : \forall_n (n \in X \to n \equiv 0 \vee \exists_{n'} ((n' \in {}^{\mathrm{co}}T_{\mathbb{N}} \vee n' \in X) \wedge n \equiv \mathrm{S}n')) \to X \subseteq {}^{\mathrm{co}}T_{\mathbb{N}}$$

and similar for the n.c. variant ${}^{\mathrm{co}}T_{\mathbb{N}}^{\mathrm{nc}}$ (with X^{nc}, $\vee^{\mathrm{nc}}$ for X, $\vee$).

Existence $\exists$, conjunction $\wedge$, disjunction $\vee$, $\vee^{\mathrm{nc}}$

These are nullary inductive predicates with parameters

$$\exists^+ \colon \forall_x (x \in P \to \exists_x (x \in P))$$
$$\exists^- \colon \exists_x (x \in P) \to \forall_x (x \in P \to C) \to C \qquad (x \notin \mathrm{FV}(C))$$

$$\wedge^+ \colon A \to B \to A \wedge B$$
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$$\vee_i^+ \colon A_i \to A_0 \vee A_1$$
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Overview

- The model $\mathcal{C}$ of partial continuous functionals (Scott, Ershov)
- TCF (theory of computable functionals)
- Realizability, soundness theorem
- Computational content of the fan theorem for coconvex bars

Kolmogorov 1932: "Zur Deutung der intuitionistischen Logik"

- Proposed to view a formula A as a computational problem, of type $\tau(A)$, the type of a potential solution or "realizer" of A.
- Example: $\forall_{n \in T_{\mathbb{N}}} \exists_{m \in T_{\mathbb{N}}} (m > n \land m \in \mathrm{Prime})$ has type $\mathbb{N} \to \mathbb{N}$.

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Type $\tau(C)$ of a c.r. predicate or formula C

$$\tau(X) := \xi \qquad (\xi \text{ uniquely assigned to } X)$$

$$\tau(\{\,\vec{x} \mid A\,\}) := \tau(A)$$

$$\tau(\underbrace{\mu_X(\forall_{\vec{x}_i}((A_{i\nu})_{\nu<n_i} \to X\vec{r}_i))_{i<k}}_{I}) := \underbrace{\mu_\xi((\tau(A_{i\nu})_{\nu<n_i}) \to \xi)_{i<k}}_{\iota_I}$$

(similar for ${}^{\mathrm{co}}I$)

$$\tau(P\vec{r}\,) := \tau(P)$$

$$\tau(A \to B) := \begin{cases} \tau(A) \to \tau(B) & (A \text{ c.r.}) \\ \tau(B) & (A \text{ n.c.}) \end{cases}$$

$$\tau(\forall_x A) := \tau(A)$$

Realizability extension $C^{\mathbf{r}}$ of c.r. predicates or formulas C

We write $z \mathbf{r} C$ for $C^{\mathbf{r}} z$ if C is a formula.

$$X^{\mathbf{r}} \quad \text{(uniquely assigned to } X\text{: } (\vec{\rho}), \text{ of arity } (\tau(X), \vec{\rho}))$$

$$\{ \vec{x} \mid A \}^{\mathbf{r}} := \{ z, \vec{x} \mid z \mathbf{r} A \}$$

$$I^{\mathbf{r}}, {}^{\mathrm{co}}I^{\mathbf{r}}$$

$$z \mathbf{r} P\vec{r} := P^{\mathbf{r}}(z, \vec{r})$$

$$z \mathbf{r} (A \to B) := \begin{cases} \forall_w (w \mathbf{r} A \to zw \mathbf{r} B) & \text{if } A \text{ is c.r.} \\ A \to z \mathbf{r} B & \text{if } A \text{ is n.c.} \end{cases}$$

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Extracted term $\mathrm{et}(M)$ of a derivation M^A with A c.r.

$$\mathrm{et}(u^A) := z_u^{\tau(A)} \quad (z_u^{\tau(A)} \text{ uniquely associated to } u^A)$$

$$\mathrm{et}((\lambda_{u^A} M^B)^{A\to B}) := \begin{cases} \lambda_{z_u}^{\tau(A)} \mathrm{et}(M) & \text{if } A \text{ is c.r.} \\ \mathrm{et}(M) & \text{if } A \text{ is n.c.} \end{cases}$$

$$\mathrm{et}((M^{A\to B} N^A)^B) := \begin{cases} \mathrm{et}(M)\mathrm{et}(N) & \text{if } A \text{ is c.r.} \\ \mathrm{et}(M) & \text{if } A \text{ is n.c.} \end{cases}$$

$$\mathrm{et}((\lambda_x M^A)^{\forall_x A}) := \mathrm{et}(M)$$

$$\mathrm{et}((M^{\forall_x A(x)} t)^{A(t)}) := \mathrm{et}(M)$$

$$\mathrm{et}(I_i^+) := \mathrm{C}_i \quad (i\text{-th constructor of } \iota \text{ associated to } I)$$

$$\mathrm{et}(I^-) := \mathcal{R}_\iota^\tau \quad (\text{recursor for } \iota)$$

$$\mathrm{et}({}^{\mathrm{co}}I^-) := \mathrm{D}_\iota \quad (\text{destructor of } \iota \text{ associated to } {}^{\mathrm{co}}I)$$

$$\mathrm{et}({}^{\mathrm{co}}I^+) := {}^{\mathrm{co}}\mathcal{R}_\iota^\tau \quad (\text{corecursor for } \iota)$$

An n.c. part of M is a subderivation with an n.c. end formula.
Such n.c. parts do not contribute to the computational content.

Theorem (Soundness)

Let M be a derivation of a formula A from assumptions $u\colon C$ (c.r.) and $v\colon D$ (n.c.) Then we can find a derivation of

$$\begin{cases} \mathrm{et}(M) \mathbf{\ r\ } A & \text{if } A \text{ is c.r.} \\ A & \text{if } A \text{ is n.c.} \end{cases}$$

from assumptions $z_u \mathbf{\ r\ } C$ and D.

Proof.

By induction on M. Few cases: $\to^{\pm}$, $\forall^{\pm}$ and c.r. axioms. □

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- View trees as sets of nodes u, v, w of type $\mathbb{L}(\mathbb{B})$ (lists of booleans), which are downward closed.
- Paths are seen as cototal objects s of type $\mathbb{S}(\mathbb{B})$.
- Sets of nodes are given by (not necessarily total) functions b of type $\mathbb{L}(\mathbb{B}) \to \mathbb{B}$. To be or not to be in b is expressed by saying that $b(u)$ is defined with 1 or 0 as its value.
- A set b of nodes is a bar if every path s hits the bar, i.e., there is an n such that $\overline{s}(n) \in b$.

For simplicity assume that bars b are upwards closed:

$$\forall_{u,p}(u \in b \to pu \in b).$$

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where $v \leq w$ means $|v| = |w|$ and v is left of w. Equivalently

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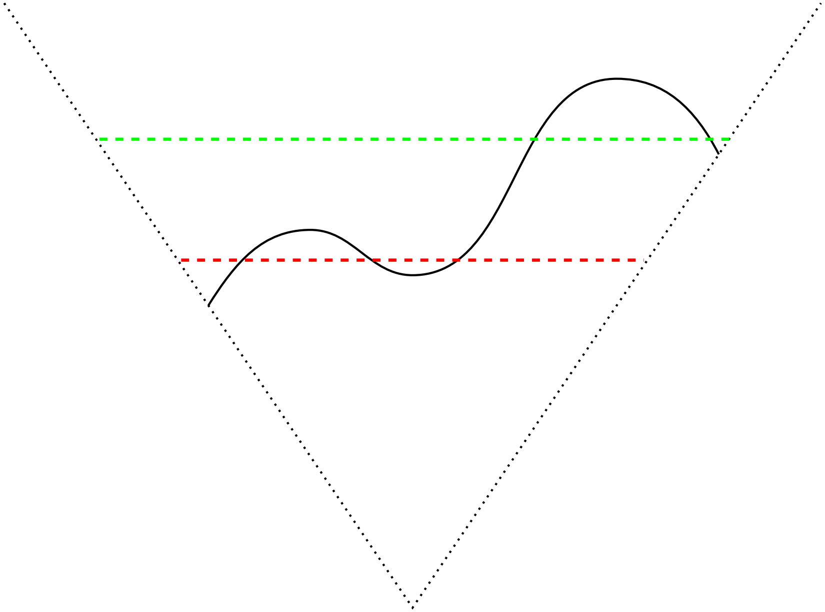

Uniform coconvexity with modulus d (direction)

- Simplification: p only, depending on node u (i.e., $p = d(u)$).
- Special case of the B&S concept. Goal: better algorithm.

Definition

A set $b \subseteq \{0,1\}^*$ is uniformly coconvex with modulus d if for all u we have: if the innermost path from pu (where $p := d(u)$) hits b in some node $v \in b$, then

$$\begin{cases} \forall_w (wpu \leq v \to wpu \in b) & \text{if } p = 0, \\ \forall_w (wpu \geq v \to wpu \in b) & \text{if } p = 1. \end{cases}$$

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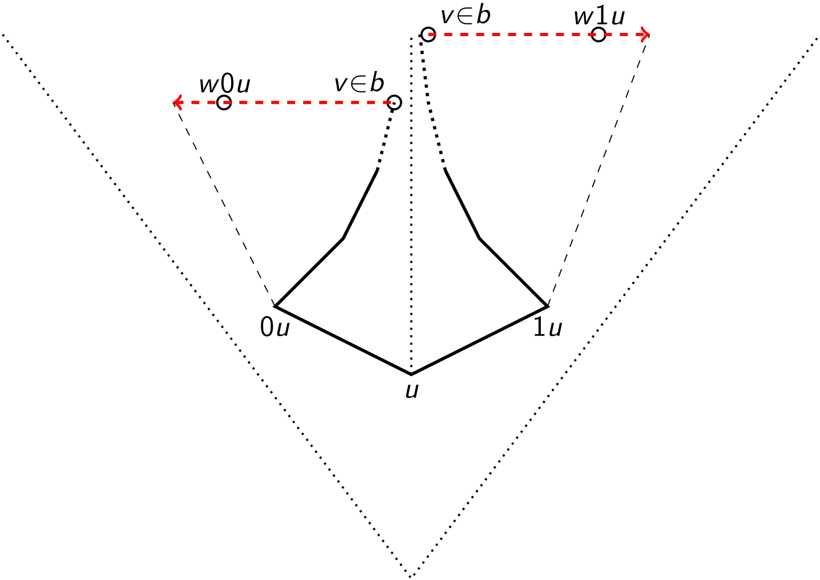
- Special case of the B&S concept. Goal: better algorithm.

Definition

A set $b \subseteq \{0,1\}^*$ is uniformly coconvex with modulus d if for all u we have: if the innermost path from pu (where $p := d(u)$) hits b in some node $v \in b$, then

$$\begin{cases} \forall_w (wpu \leq v \to wpu \in b) & \text{if } p = 0, \\ \forall_w (wpu \geq v \to wpu \in b) & \text{if } p = 1. \end{cases}$$



v∈b
w1u
w0u
v∈b
0u
1u
u

Recall: ${}^{\mathrm{co}}T_{\mathbb{S}(\rho)}$ is the greatest fixed point of the clause

$$s \in {}^{\mathrm{co}}T_{\mathbb{S}(\rho)} \to \exists_{x \in T_\rho, s' \in {}^{\mathrm{co}}T_{\mathbb{S}(\rho)}} (s = xs')$$

The corecursion operator ${}^{\mathrm{co}}\mathcal{R}^{\tau}_{\mathbb{S}(\rho)}$, of type

$$\tau \to (\tau \to \rho \times (\mathbb{S}(\rho) + \tau)) \to \mathbb{S}(\rho)$$

is defined by

$${}^{\mathrm{co}}\mathcal{R}xf = \begin{cases} yz & \text{if } f(x) = \langle y, \mathrm{InL}(z) \rangle, \\ y({}^{\mathrm{co}}\mathcal{R}x'f) & \text{if } f(x) = \langle y, \mathrm{InR}(x') \rangle. \end{cases}$$

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Lemma (Cototality of corecursion)

Let $f : \tau \to \rho \times (\mathbb{S}(\rho) + \tau)$ *be of* InR*-form, i.e.,* $f(x)$ *has the form* $\langle y, \mathrm{InR}(x') \rangle$ *for all* x*. Then* ${}^{\mathrm{co}}\mathcal{R}xf \in {}^{\mathrm{co}}T_{\mathbb{S}(\rho)}$ *for all* x*.*

Proof.

By coinduction with competitor predicate

$$X := \{\, z \mid \exists_x (z = {}^{\mathrm{co}}\mathcal{R}xf) \,\}.$$

Need to prove that X satisfies the clause defining ${}^{\mathrm{co}}T_{\mathbb{S}(\rho)}$:

$$\forall_z (z \in X \to \exists_y \exists_{z'} (z' \in X \land z = yz')).$$

Let $z = {}^{\mathrm{co}}\mathcal{R}xf$ for some x. Since f is assumed to be of InR-form we have y, x' such that $f(x) = \langle y, \mathrm{InR}(x') \rangle$. By the definition of ${}^{\mathrm{co}}\mathcal{R}^{\tau}_{\mathbb{S}(\rho)}$ we obtain ${}^{\mathrm{co}}\mathcal{R}xf = y({}^{\mathrm{co}}\mathcal{R}x'f)$. Use ${}^{\mathrm{co}}\mathcal{R}x'f \in X$. □

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The escape path $s_d \in \mathbb{S}(\mathbb{B})$ is constructed from d corecursively:

Start with the root node. At any node u, take the opposite direction to what $d(u)$ says, and continue.

Definition (Distance)

$$D_b n u := \forall_v (|v| = n \to vu \in b)$$

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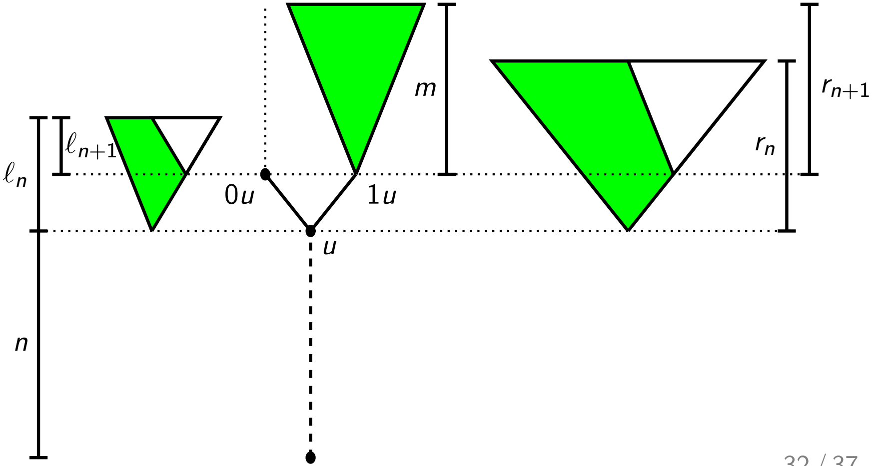
Lemma (BoundL, BoundR)

Let b be a uniformly coconvex bar with modulus d. Then for every n there are bounds ℓ_n, r_n for the b-distances of all nodes of the same length n that are left / right of $\overline{s_d}(n)$.

Proof. For $n = 0$ there are no such nodes.

Consider $\overline{s_d}(n+1) = (s_d)_n u$ of length $n+1$. Assume $(s_d)_n = 0$. Then every node to the left of $0u$ is a successor node of one to the left of u, and hence $\ell_{n+1} = \ell_n - 1$. The nodes to the right of $0u$ are $1u$ and then nodes which are all successor nodes of one to the right of u. Since $1u$ is $d(u)u$, by assumption we have its b-distance m. Let $r_{n+1} = \max(m, r_n - 1)$.



Extracted term for `BoundL`

```
[hit,d,n](Rec nat=>nat)n 0
 ([n0,n1][case (d(U d n0))
   (True -> Pred n1 max hit(True::U d n0)cCoSTConstFalse)
   (False -> Pred n1)])
```

and for `BoundR`

```
[hit,d,n](Rec nat=>nat)n 0
 ([n0,n1][case (d(U d n0))
 (True -> Pred n1)
 (False -> Pred n1 max hit(False::U d n0)cCoSTConstTrue)])
```

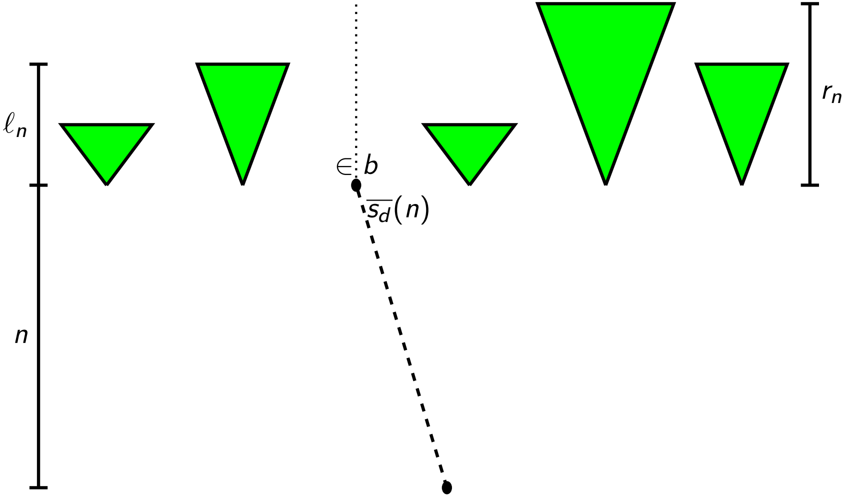
with `hit` of type $\mathbb{L}(\mathbb{B}) \to \mathbb{I} \to \mathbb{N}$.

Theorem

Let b be a uniformly coconvex bar with modulus d. Then b is a uniform bar, i.e.,

$$\exists_m \forall_u (|u| = m \to u \in b).$$

Let s_d be the escape path. Since b is a bar, the escape path s_d hits b at some length n. Use lemma Bounds: the uniform bound is $n + \max(\ell_n, r_n)$



Extracted term

```
[hit,d]
 cBoundL hit d(hit Nil(cEscCoST d))max
 cBoundR hit d(hit Nil(cEscCoST d))+
 hit Nil(cEscCoST d)
```

with `hit` of type $\mathbb{L}(\mathbb{B}) \to \mathbb{I} \to \mathbb{N}$.

Reference

Josef Berger and Gregor Svindland,
Constructive convex programming.
To appear: Proof-Computation – Digitalization in Mathematics, Computer Science and Philosophy (eds. Mainzer. Schuster, S.) World Scientific, Singapore, 2018