Computable dyadic subbases

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    - ★ Every second countable space $(X, (B_n)_{n\in\mathbb{N}})$ can be embedded into $\mathbb{S}^\omega$ by $\varphi(x)(n) = 1$ iff $x \in B_n$.

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    - ★ Enumeration-based $\{0,1\}^\omega$-representation $\psi_{\mathbb{S}^\omega} :\subseteq \{0,1\}^\omega \to \mathbb{S}^\omega$. A $\mathbb{S}^\omega$-representation ψ induces a $\{0,1\}^\omega$-representation $\psi_{\mathbb{S}^\omega} \circ \psi$.

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0 1

$\backslash$ $/$

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- Why do we study $\mathbb{T}^\omega$-representation?

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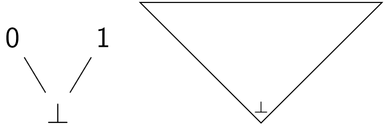
  - Every n-dimensional second countable metrizable space can be embed into $\mathbb{T}^\omega_n$, which is a subspace of $\mathbb{T}^\omega$ with up to n copies of $\perp$ [T 2002].
- Order structures $(\mathbb{T}, \preceq)$ and $(\mathbb{T}^\omega, \preceq)$.
  - Natural representation of a space with order.



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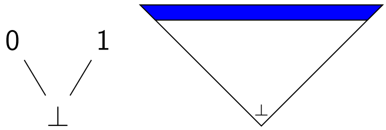
- Contains $\{0,1\}^\omega$ as top elements.
  - Sub-structure of the space can be represented with $\{0,1\}^\omega$.



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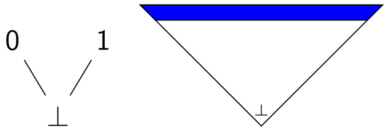
- A bottomed sequence is an unspecified sequence.
  - $10\perp10.. = \{10010.., 10110...\}$.



Matching-representation

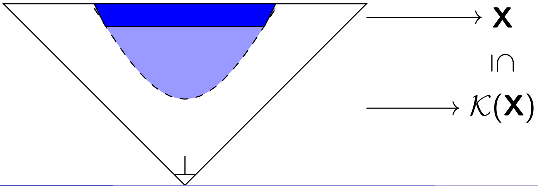
$\mathcal{K}(\mathbf{X})$: the set of compact subsets of a topological space $\mathbf{X}$.

Definition

A matching representation is a pair of a representation $\delta :\subseteq \{0,1\}^\omega \to \mathbf{X}$ and an order-preserving $\mathbb{T}^\omega$-representation $\psi :\subseteq \mathbb{T}^\omega \to \mathcal{K}(\mathbf{X}) \setminus \{\emptyset\}$ with an upper-closed domain such that

$$\psi(p) = \{\delta(q) \mid p \preceq q \in \{0,1\}^\omega\}.$$

- Furthermore, if $\psi(p) = A$ and A is a finite set, then the number of bottoms in p is exactly $|A| - 1$.
- cf. domain representation[Blanck 2000].



Represented Space [P 2015]

- A-represented space $\mathbf{X} = (X, \delta_X)$ is a pair of a set X and a partial surjection $\delta_X :\subseteq A \to X$.
- We say that two $(\{0,1\}, \mathbb{T}, \mathbb{S}, \mathbb{N}^\omega)$- represented spaces are computably isomorphic if the conversion of the names is computable.
- For represented spaces $\mathbf{X}$ and $\mathbf{Y}$, we define represented spaces
  - $\mathcal{C}(\mathbf{X}, \mathbf{Y})$: the space of continuous functions from $\mathbf{X}$ to $\mathbf{Y}$.
  - $\mathcal{O}(\mathbf{X}) (= \mathcal{C}(\mathbf{X}, \mathbb{S}))$: the space of open subsets of $\mathbf{X}$.
  - $\mathcal{A}(\mathbf{X})$: the space of closed subsets of $\mathbf{X}$ (negative information).
  - $\mathcal{V}(\mathbf{X})$: the space of closed subsets of $\mathbf{X}$ (positive information), which we call overt sets.
  - $\mathcal{K}(\mathbf{X})$: the space of compact subsets of $\mathbf{X}$.
- Our goal: given a represented space $\mathbf{X}$, construct a matching representation $(\delta :\subseteq \{0,1\}^\omega \to \mathbf{X},\ \psi :\subseteq \mathbb{T}^\omega \to \mathcal{K}(\mathbf{X}) \setminus \{\emptyset\})$ which are computably isomorphic to the given $\mathbf{X}$ and the $\mathcal{K}(\mathbf{X}) \setminus \{\emptyset\}$.

The Theorem

Theorem

Every computably compact computable metric space X admits matching representations of X and $\mathcal{K}(\mathbf{X}) \setminus \{\emptyset\}$.

- A computable metric space (X, d, α) is a separable metric space (X, d) with some computable structure. (We give the definition later.)
- A computable metric space has the Cauchy representation δ_X and we consider the represented space $\mathbf{X} = (X, \delta_X)$.
- $\mathbf{X}$ is computably compact: isEmpty : $\mathcal{A}(\mathbf{X}) \to \mathbb{S}$ is computable.
- We can compute a matching representation from the structure of a computably compact computable metric space.
- This theorem has applications to finite closed choice and Weihrauch reducibility.

The Procedure

Computably compact computable metric space

$\Downarrow$

proper computable dyadic subbase

$\Downarrow$

Pruned-tree representation

$\Downarrow$

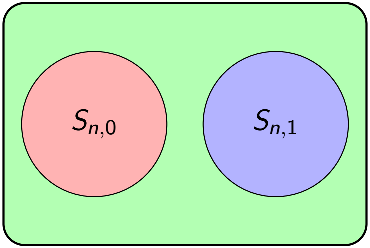
Matching representation

Computable dyadic subbase

Definition ([T 2004])

A *dyadic subbase* over a set X is a map $S : \mathbb{N} \times \{0,1\} \to \mathcal{P}(X)$ such that $S_{n,0} \cap S_{n,1} = \emptyset$ for every $n \in \mathbb{N}$ and
if $\{(n,i) \mid x \in S_{n,i}\} = \{(n,i) \mid y \in S_{n,i}\}$ for $x, y \in X$, then $x = y$.

- $S_{n,\perp} = X \setminus (S_{n,0} \cup S_{n,1})$.
- $\varphi_S(x)(n) = \begin{cases} 0 & (x \in S_{n,0}), \\ 1 & (x \in S_{n,1}), \\ \perp & (x \in S_{n,\perp}). \end{cases}$
- $\varphi_S : X \hookrightarrow \mathbb{T}^\omega$: embedding into $\mathbb{T}^\omega$.
- $\mathbf{X}_S = (X, \varphi_S^{-1})$ is an admissible $\mathbb{T}^\omega$-represented space.
- We say that S is a computable dyadic subbase of a represented space $\mathbf{X}$ if $\mathbf{X}_S$ is computably isomorphic to $\mathbf{X}$.



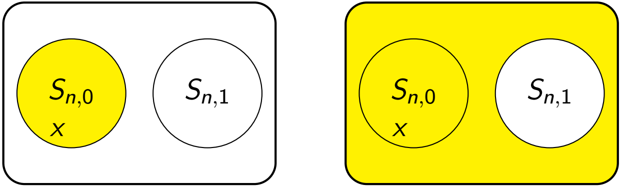
n-th coordinate

Two kinds of informations.

- Each finite sequence $p \in \mathbb{T}^*$ specifies

$$S(p) = \bigcap_{n \in \mathrm{dom}(p)} S_{n,p(n)}$$

$$\bar{S}(p) = \bigcap_{n \in \mathrm{dom}(p)} (X \setminus S_{n,1-p(n)}) = \bigcap_{n \in \mathrm{dom}(p)} (S_{n,p(n)} \cup S_{n,\perp})$$



Example: For $p = 0\perp10$, $S(p) = S_{0,0} \cap S_{2,1} \cap S_{3,0}$ and $\bar{S}(p) = X \setminus (S_{0,1} \cup S_{2,0} \cup S_{3,1})$.

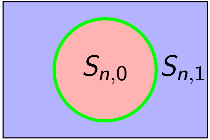
- $\{S(p) \mid p \in \mathbb{T}^*\}$ is the base generated by the subbase $\{S_{n,0}, S_{n,1} \mid n \in \mathbb{N}\}$.

Proper dyadic subbase

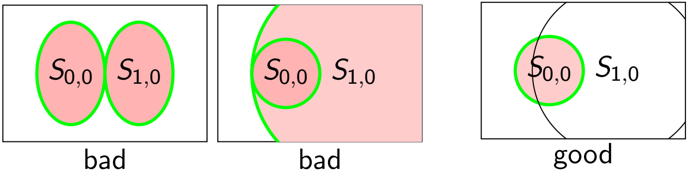
Definition

A dyadic subbase S is *proper* if $\bar{S}(p) = \mathrm{cl}\, S(p)$ for every $p \in \mathbb{T}^*$.

- Generalization of Gray-code.
- $S_{n,0}$ and $S_{n,1}$ are exteriors of each other. (The case $p = \perp^n 1$.) $S_{n,\perp}$ is the common boundary.



- $S_{0,\perp}$ and $S_{1,\perp}$ do not touch. (The case $p = 00$.)



Computability notions of $\bar{S}(p)$ and $\mathrm{cl}\, S(p)$

Two computability notions $\mathcal{A}(\mathbf{X})$ and $\mathcal{V}(\mathbf{X})$ for closed sets.

- $\bar{S}(p) \in \mathcal{A}(\mathbf{X})$ because $\bar{S}(p) = X \setminus (\bigcup_{n \in \mathrm{dom}(p)} S_{n,1-p(n)})$.
  - $A \in \mathcal{A}(\mathbf{X}) \iff A^C \in \mathcal{O}(\mathbf{X})$.
  - Representation by negative information.
  - $\mathcal{A}(\mathbf{X})$ and $\mathcal{K}(\mathbf{X})$ computably isomorphic if $\mathbf{X}$ is computably compact Hausdorff spaces.
  - For a continuous function f and $A \in \mathcal{K}(\mathbf{X})$, maximum value of $f(A)$ approximated from above.
- $\mathrm{cl}\, S(p) \in \mathcal{V}(\mathbf{X})$ because $\mathrm{cl}\, S(p) = \mathrm{cl}\,(\bigcap_{n \in \mathrm{dom}(p)} S_{n,p(n)})$.
  - $A \in \mathcal{V}(\mathbf{X})$ is represented by enumeration of $\{U \mid U \cap A \neq \emptyset\}$.
  - Representation by positive information.
  - For a continuous function f and $A \in \mathcal{V}(\mathbf{X})$, maximum value of $f(A)$ approximated from below.
- If S is a proper dyadic subbase, then $\bar{S}(p) \in \mathcal{V}(\mathbf{X}) \wedge \mathcal{K}(\mathbf{X})$.
  - Maximum value of $f(A)$ computable.

Exact subsets

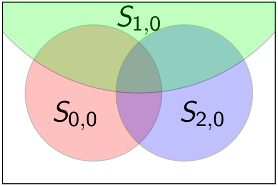
$$S(p) = \bigcap_{n \in \mathrm{dom}(p)} S_{n,p(n)}$$

$$\bar{S}(p) = \bigcap_{n \in \mathrm{dom}(p)} (X \setminus S_{n,1-p(n)}) = \bigcap_{n \in \mathrm{dom}(p)} \mathrm{cl}\, S_{n,p(n)}.$$

$$S_{\mathrm{ex}}(p) = \bigcap_{n<|p|} S_{n,p(n)},$$

$$\bar{S}_{\mathrm{ex}}(p) = \bigcap_{n<|p|} \mathrm{cl}\, S_{n,p(n)}.$$

$p = 0\bot 0$



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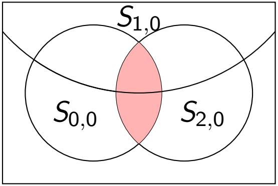

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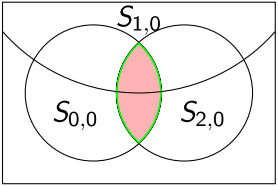

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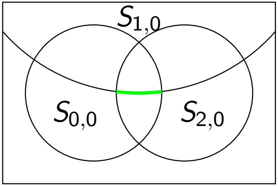

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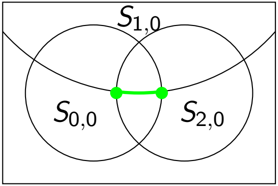


$\bar{S}_{\mathrm{ex}}(p) \in \mathcal{A}(X)$ if S is proper.

Definition

Let S be a dyadic subbase of a space $\mathbf{X}$. Define

$$\widehat{K}_S = \{e \in \mathbb{T}^* \mid \bar{S}_{\mathrm{ex}}(e) \neq \emptyset\} \ (\subseteq \mathbb{T}^*),$$

$$\widehat{D}_S = \text{the ideal completion of } \widehat{K}_S \ (\subseteq \mathbb{T}^\omega), \quad \widehat{L}_S = \widehat{D}_S \setminus \widehat{K}_S.$$

Theorem

Suppose that S is a proper computable dyadic subbase of a computably compact Hausdorff X.

1. $\bar{S}_{\mathrm{ex}}(e) = \emptyset$ *is semi-decidable, and therefore* $\widehat{K}_S$ *is r.e.*
2. $\widehat{K}_S$ *is finitely branching (i.e.,* $\{e \mid d \prec^1 e\}$ *is finite for* $\forall d \in \mathbb{T}^*$*).*
3. $\varphi_S(X) \subseteq \widehat{L}_S$ *is the set of minimal elements of* $\widehat{L}_S$*. Moreover,* X *is a retract of* $\widehat{L}_S$*. Therefore, every infinite path* $e_0 \prec^1 e_1 \prec^1 \ldots$ *in* $\widehat{K}_S$ *identifies a unique point* x*. [T, Tsukamoto 2015].*

These properties are used to expand $\widehat{K}_S$ to a tree, and then form a matching representation of X.

The Procedure

Computably compact computable metric space

$\Downarrow$

Proper computable dyadic subbase

$\Downarrow$

Pruned-tree representation

$\Downarrow$

Matching representation

Non-effective version

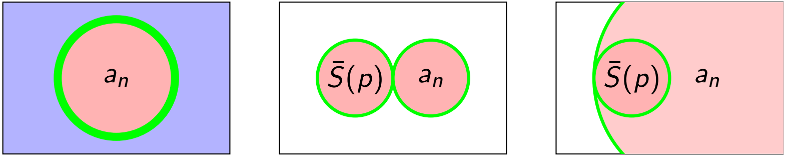
Theorem

Every separable metric space has a proper dyadic subbase.

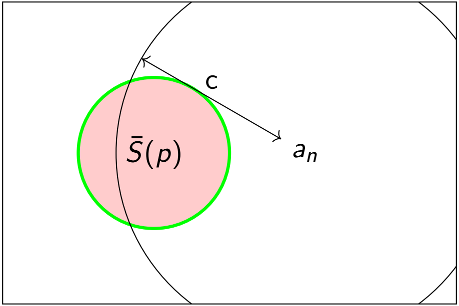
- Proved in [Ohta, Yamada, T 2011] for a special case and in [Ohta, Yamada, T 2013] for the full case.
- Tsukamoto gave an elegant proof in [Tsukamoto 2017]. (In that paper, he also proved that every locally compact separable metric space has a strongly proper dyadic subbase.)
- We effectivize his proof and show that every computably compact computable metric space has a computable proper dyadic subbase.

Tsukamoto's idea:

- Choose appropriate $a_n \in X$ and $c_n \in \mathbb{R}^{>0}$ and define $S_{n,0} = \{x \mid d(x, a_n) < c_n\}$, $S_{n,1} = \{x \mid d(x, a_n) > c_n\}$.
- In order that they form a (sub)base, consider a dense subset $(b_i)_{i\in\mathbb{N}}$ of X and a base $(U_j)_{j\in\mathbb{N}}$ of $\mathbb{R}^{>0}$, and for $n = \langle i, j\rangle$, set $a_n = b_j$ and choose c_n from U_j .
- In order that it is proper, avoid (1) boundary $\{x \mid d(x, a_n) = c_n\}$ has an interior and (2) for every p with $|p| = n$, boundaries do not touch.



- $c \in \mathbb{R}$ is a local maximum of a continuous function $f : X \to \mathbb{R}$ if c is the maximum value of $f|_V$ for some open subset V. Local maximum and local minimum values are called local extrema.
- In the above cases, c_n is a local extrema of $f(x) = d(x, a_n)$ restricted to $\bar{S}(p)$.



- Choose $c \in \mathbb{R}$ which is not a local extrema of $f(x) = d(x, a_n)$ restricted to $\bar{S}(p)$ for every $p \in \{0, 1, \perp\}^n$.
- Then, define $S_{n,0} = \{x \mid d(x, a_n) < c\}$ and $S_{n,1} = \{x \mid d(x, a_n) > c\}$.
- Since an extrema of f is a maximum (or minimum) value of $f|_V$ for some open subset V, There are countably many local extrema for a countably based space. Therefore, we can avoid them to choose c.

Effectivization of the proof

Definition

1. A computable metric space (X, d, α) is a separable metric space (X, d) with a dense sequence $\alpha : \mathbb{N} \to X$ such that $d \circ (\alpha \times \alpha) : \mathbb{N}^2 \to \mathbb{R}$ is a computable double sequence of real numbers.
2. We define the Cauchy representation $\delta_X :\subseteq \mathbb{N}^{\mathbb{N}} \to X$.

Theorem

Every computably compact computable metric space admits a proper computable dyadic subbase.

- We cannot use the cardinality argument to choose c_n.
- We use the Computable Baire category theorem [Brattka 2001].

Theorem (Computable Baire category theorem [Brattka 2001])

There exists a computable operation $\Delta :\subseteq \mathcal{A}(\mathbf{X})^{\mathbb{N}} \times \mathcal{O}(\mathbf{X}) \rightrightarrows \mathbf{X}$ such that, for any sequence $(A_n)_{n\in\mathbb{N}}$ of closed nowhere dense subsets of X and a non-empty open subset I, $\Delta((A_n)_{n\in\mathbb{N}}, I) \in I \setminus \cup_{n=0}^{\infty} A_n$.

- Apply this to the case $\mathbf{X}$ is $\mathbb{R}$.
- We need to represent the set of local extrema of f as an element of $\mathcal{A}(\mathbf{X})^{\mathbb{N}}$.
- Recall that $f(x)$ is $d(x, a)$ restricted to $A = \bar{S}(p)$ for each $|p| = n$.
- The maximum value of f on $A \in \mathcal{K}(\mathbf{X})$ is computable because $A \in \mathcal{K}(\mathbf{X}) \wedge \mathcal{V}(\mathbf{X})$.
- However, we want to compute not maximum value but local maximum values of f on A. They are maximum values of $f|_{A\cap V}$ for open V.

- We denote by $\bar{M}(a,r)$ the maximum value of f on $A \cap \bar{B}(a,r)$.
- We denote by $M(a,r)$ the maximum value of f on $A \cap \mathrm{cl}\, B(a,r)$.
- For some $a \in \mathbf{X}$ and $r \in \mathbb{Q}^{>0}$, a local maximum value of f on A is, at the same time, $\bar{M}(a,r)$ and $M(a,2r)$.
- Let $D(a,r) = \{x : M(a,2r) \leq x \leq \bar{M}(a,r)\}$. Since $M(a,2r) \geq \bar{M}(a,r)$ in general, it is a one-point set or an empty set. It is a one-point set iff it is a local maximum value.
- $A \cap \bar{B}(a,r) \in \mathcal{A}(\mathbf{X})$ and thus $\bar{M}(a,r)$ is approximated from above
- $A \cap \mathrm{cl}\, B(a,2r) \in \mathcal{V}(\mathbf{X})$ and thus $M(a,2r)$ is approximated from below.
- Thus, $D(a,r)$ is approximated from above and below, and thus $D(a,r) \in \mathcal{A}(\mathbf{X})$.
- Now, consider all $D(a,r)$ for $a \in \alpha$ and $r \in \mathbb{Q}^{>0}$, and apply the computable Baire category theorem.

Conclusion

- $\mathbb{T}^\omega$-representation and Matching representation.
- Proper computable dyadic subbases.
- Every computably compact computable metric space admits a proper computable dyadic subbase.

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