Some results on the computational complexity of noisy dynamical systems

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Joint work with Jon Schneider

University of Tokyo

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We are interested in statistical properties of the system for $t \to \infty$.

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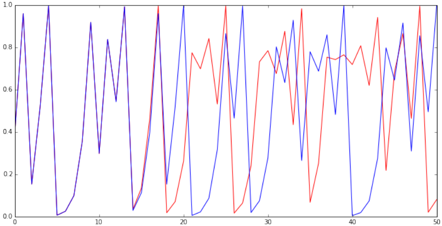

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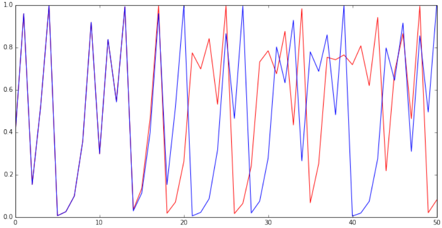

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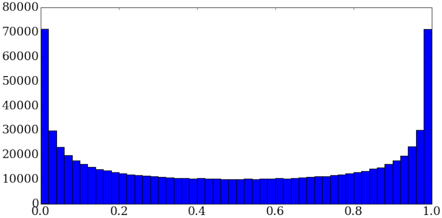

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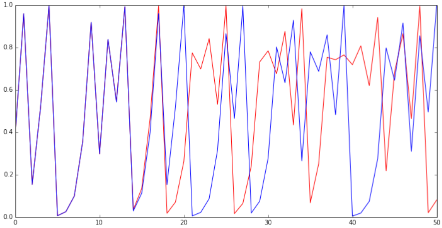

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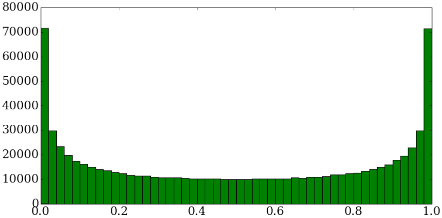

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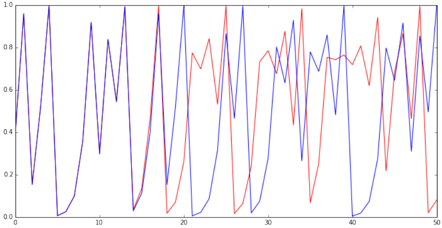

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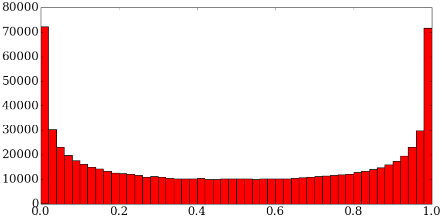

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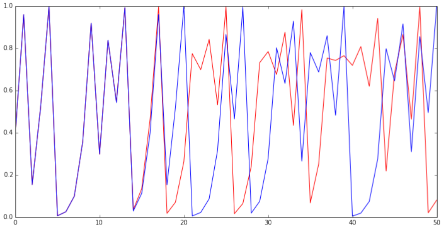

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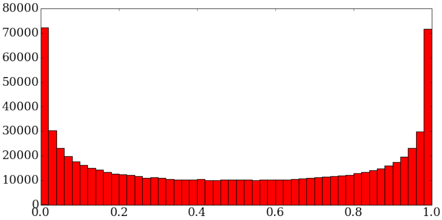

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It can be shown that the limit density is $p(x) = \frac{1}{\pi\sqrt{x(1-x)}}$.

Ergodic Theory

Definition

Let Φ_t be a dynamical system and X a measurable space. A probability measure μ is **invariant** for Φ_t if

$$\mu(\Phi_{-t}(A)) = \mu(A)$$

for all $t \in T$ and all measurable sets A.

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Definition

An invariant measure is called **ergodic** if

$$\Phi_{-t}(A) = A \text{ for all } t \in T \Rightarrow \mu(A) = 0 \vee \mu(A) = 1$$

Computable measures

Definition (Computable metric space)

A **computable metric space** is a triple (X, d, S) where

- (X, d) is a separable complete metric space
- $S = \{s_i : i \in \mathbb{N}\}$ is a countable dense subset
- The reals $d(s_i, s_j)$ are uniformly computable in i, j

Elements of S are called **ideal points**.

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Definition

We say a probability measure is **computable** if it is a computable point in $\mathcal{M}(X)$.

Computability of Invariant Measures

Theorem (Gelatolo, Hoyrup, Rojas)

There is a computable map $T : [0,1] \to [0,1]$ *such that the system* $([0,1], T)$ *has an ergodic measure that is not computable.*

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Definition

We say a probability measure is polynomial-time computable if the function $F(x) = \mu([0, x])$ is polynomial-time computable.

Dynamical systems with noise

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P_t has to fulfill some properties:

1. For each measurable $A \subseteq X$, the function $(t, x) \mapsto P_t(A|x)$ is measurable.
2. For each t, x, $A \mapsto P_t(A|x)$ is a probability measure.
3. For any $s, t \in T$ it is $P_{s+t}(A|x) = \int_X P_t(dy|x) P_s(A|y)$.

This defines a Markov process $(x_t)_{t \in T}$.

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Definition

Given $\mu \in \mathcal{M}(X)$ the **push forward** is defined by

$$(S_t \mu)(A) = \int_X P_t(A|x) \, \mathrm{d}\mu$$

μ is invariant if $S_t \mu = \mu$ for all $t > 0$.

Discrete-time dynamical systems with noise

Definition

For $\varepsilon > 0$ consider a family of Borel probability measures $(Q_x^\varepsilon)_{x \in X}$ over X. A **random perturbation** S_ε of the dynamic system (X, T) is a Markov Chain X_t with transition probabilities

$$P(A|x) = P\{X_{t+1} \in A : X_t = x\} = Q_{f(x)}^\varepsilon(A)$$

for any $x \in X$ and Borel set $A \subseteq M$.

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Theorem (Braverman, Grigo, Rojas)

Let $S = (X, T)$ be a computable system over a compact subset $X \subseteq \mathbb{R}^d$. Assume Q_x^ε is uniform on the ε-ball around x. Then, for almost every $\varepsilon > 0$, the ergodic measures of S_ε are all computable.

Complexity of noisy systems

Previous results (mainly by Braverman, Grigo, Rojas and Schneider) can be summarized as follows:
It is assumed that the noise is Gaussian.

TIME	upper bound	lower bound
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Additionally there are complexity bounds in the noise-parameter ϵ.

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A function $\varphi \in \#\mathcal{P}$ if there is some set $A \in \mathcal{P}$ and a polynomial $p : \mathbb{N} \to \mathbb{N}$ such that for all $u \in \Sigma^*$, $\varphi(u) = \#\{w \in \Sigma^{p(|u|)} \, : \, uw \in A\}$.

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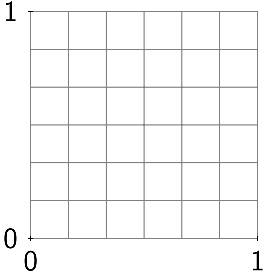
- $f_u(x)$ can be approximated to precision 2^{-n} in time $poly(n + |u|)$.
- It suffices to approximate $\mu_{f_u}([0, 1/3])$ with precision $poly(|u|)$ to extract $\varphi(u)$.

Construction of f_u



$\varphi \in \#\mathcal{P}$, $u \in \Sigma^*$, $A \in P$ "witness function" for φ, $p : \mathbb{N} \to \mathbb{N}$ length of witness.
μ: invariant measure for $S_{f_u, \frac{1}{6}}$. Want $\mu([0, 1/3]) \widehat{=} \varphi(u)$.

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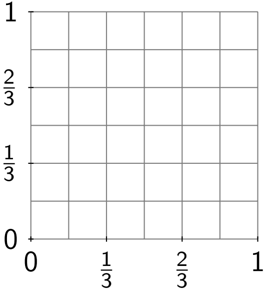

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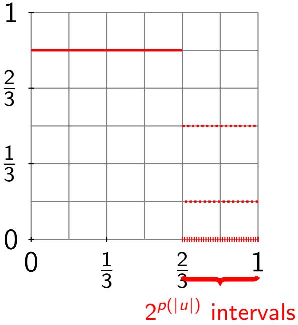

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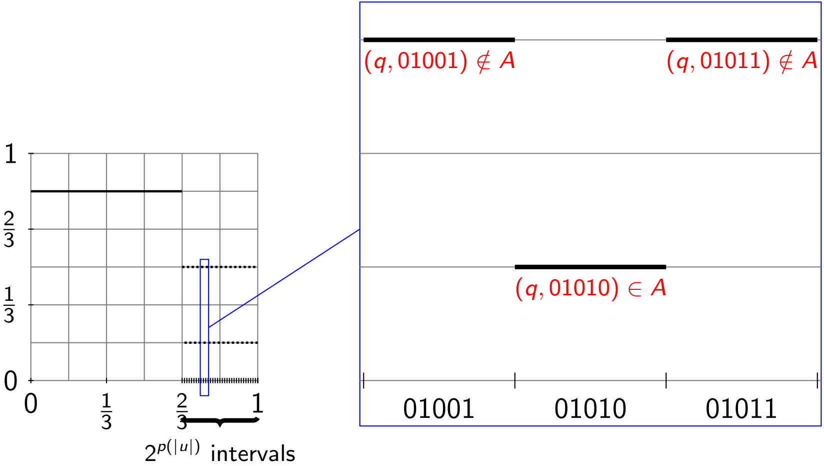

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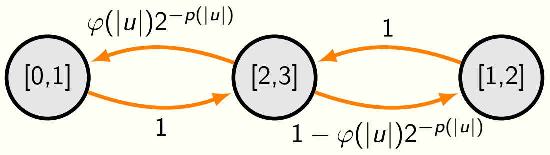
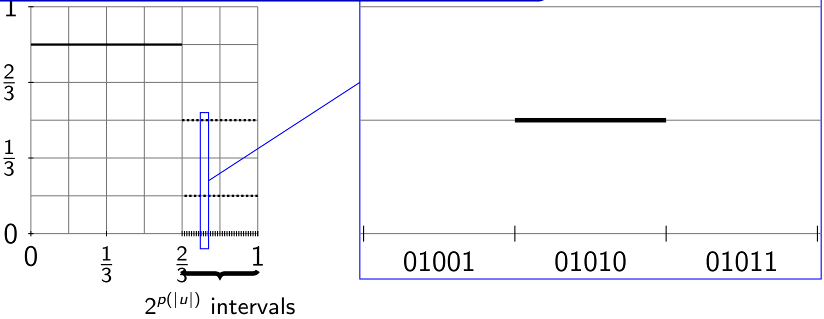
Markov chain





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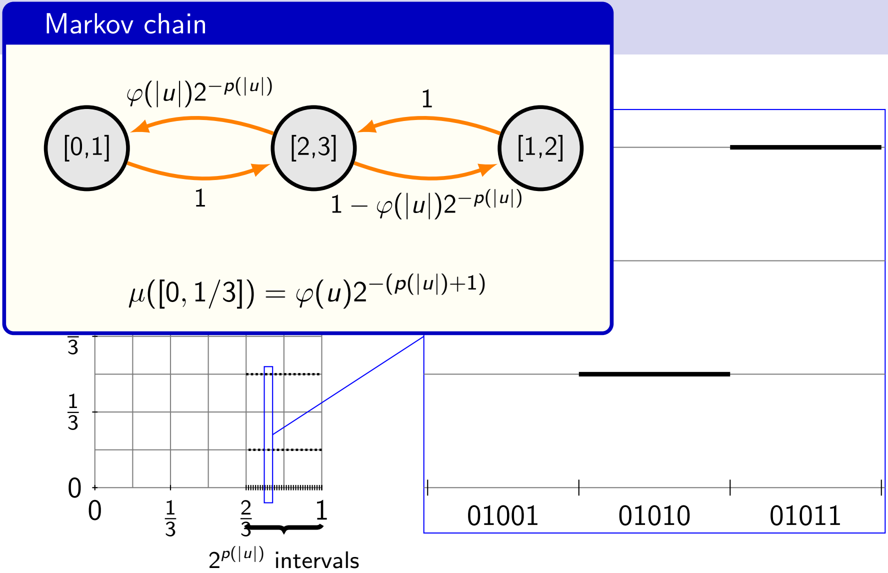

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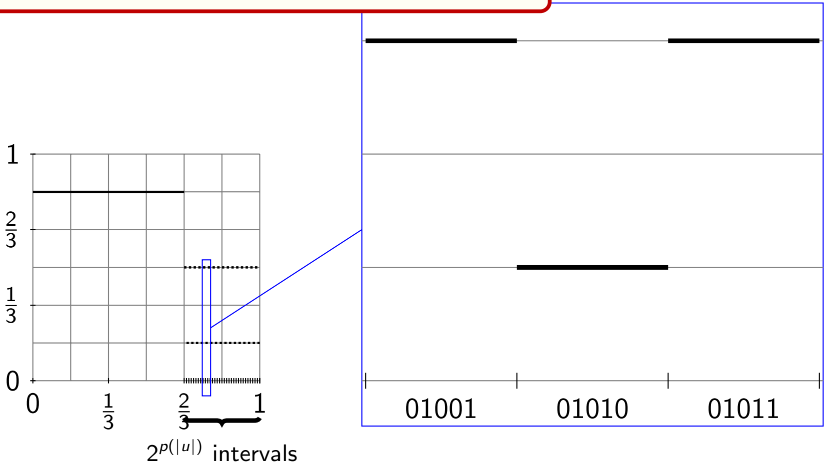
Note

f not continuous $\Rightarrow$ not computable!



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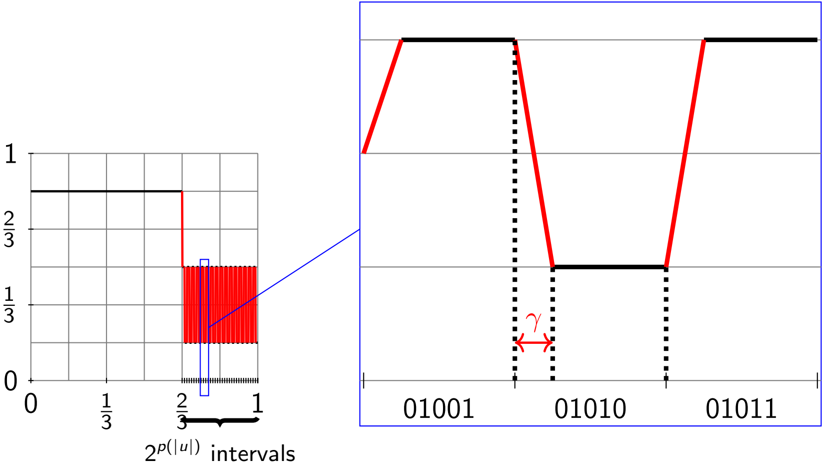

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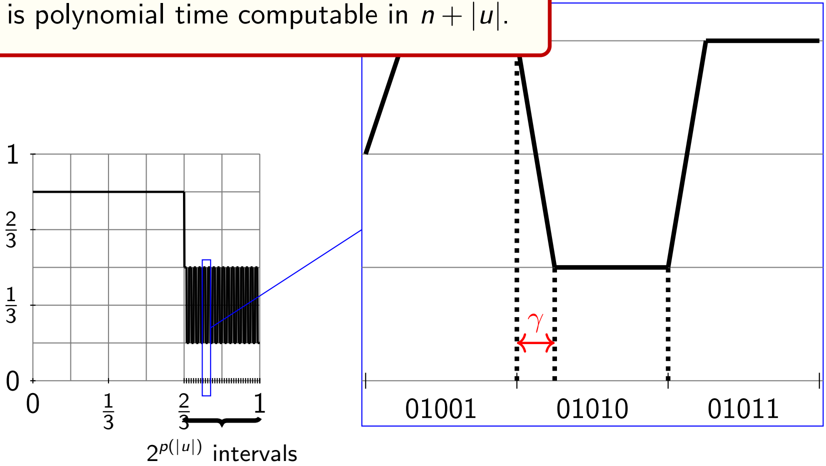
Note

Let $\gamma = 2^{-2p(|u|)-5}$.
f is polynomial time computable in $n + |u|$.



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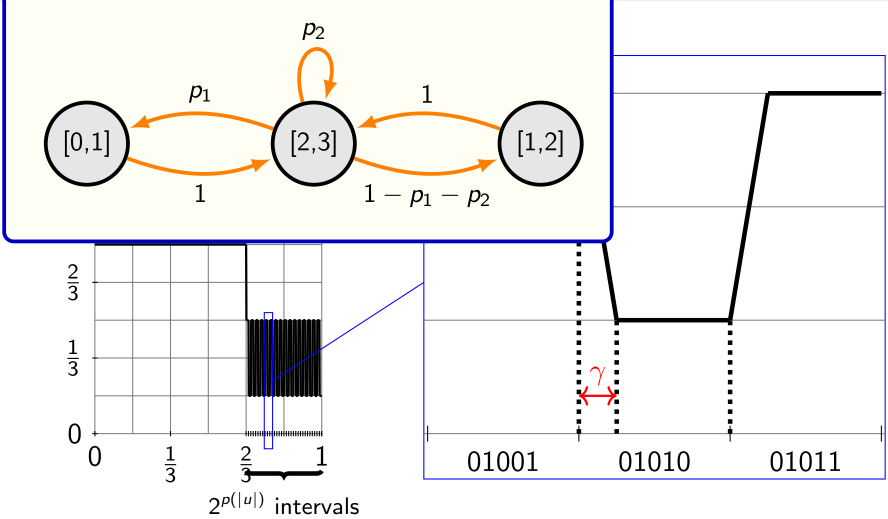

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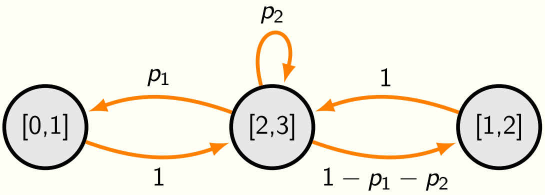
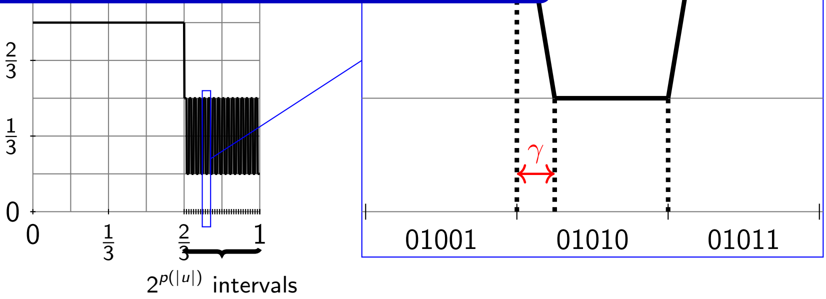
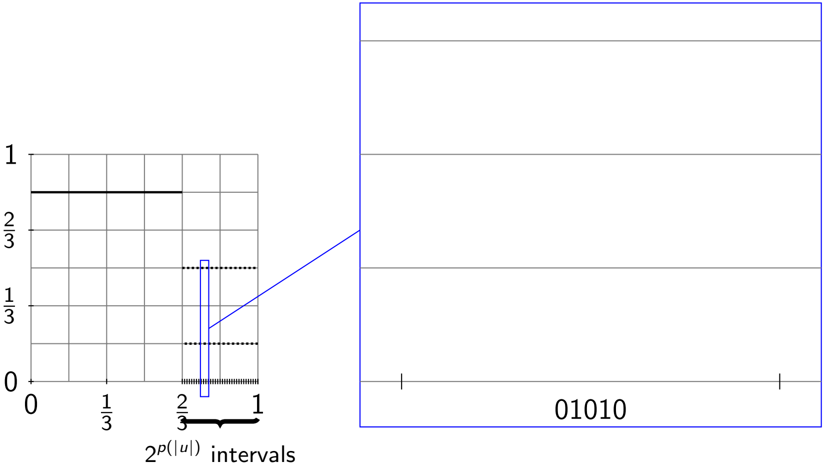




What about Gaussian noise?

Construction for Gaussian noise



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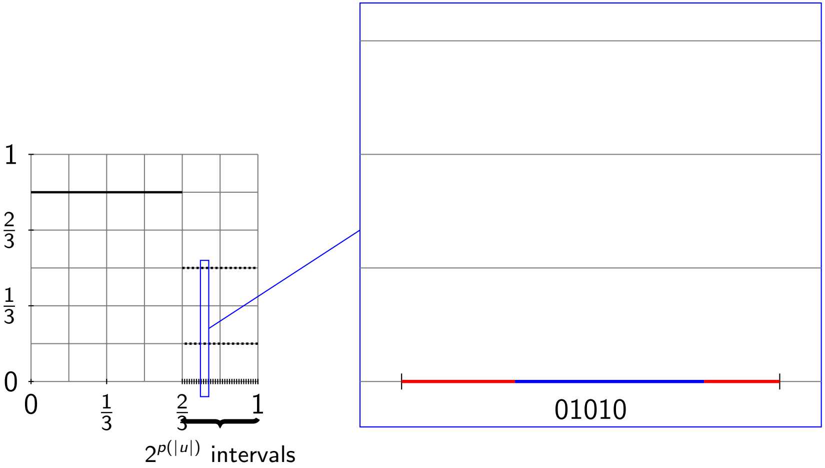
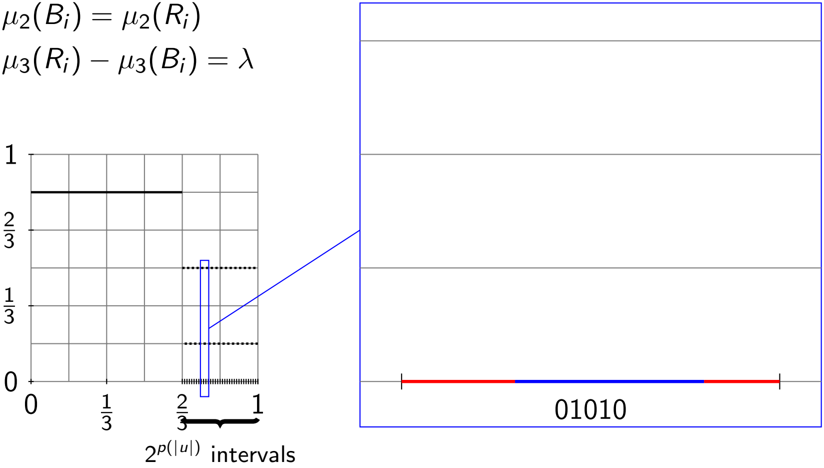


Construction for Gaussian noise

In each Region we want:

- $\mu_1(B_i) = \mu_1(R_i)$
- $\mu_2(B_i) = \mu_2(R_i)$
- $\mu_3(R_i) - \mu_3(B_i) = \lambda$



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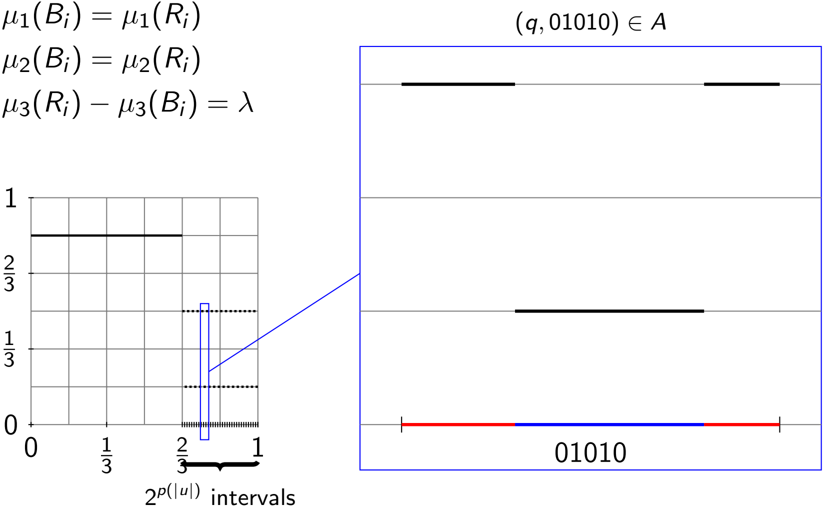

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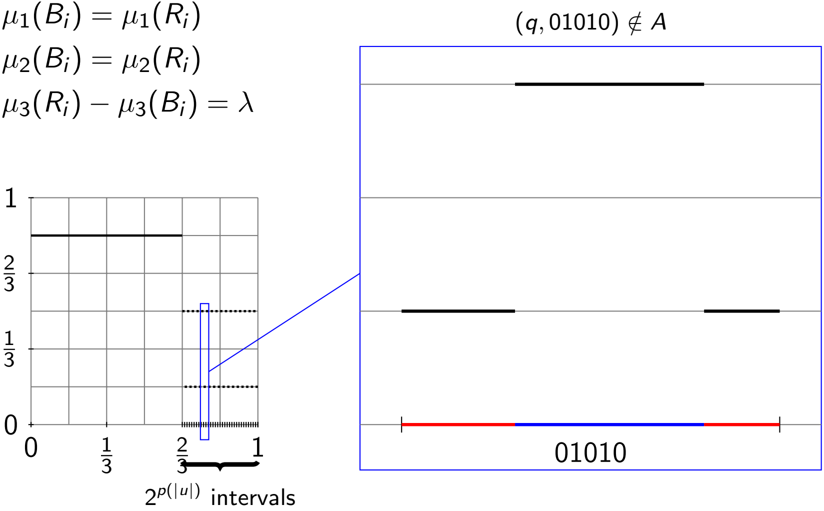
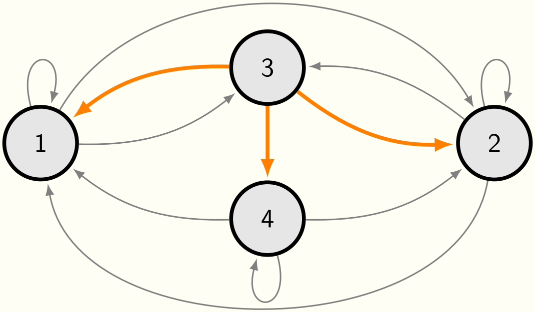


Construction for Gaussian noise

In each Region we want:

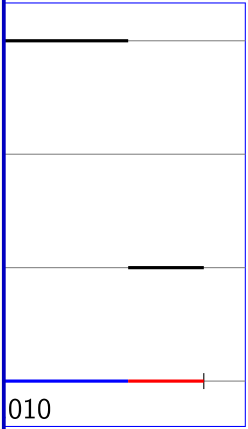
Markov chain



Constants computable from the Gaussians.

Functions depending on $\varphi(u)$.

010) $\notin A$



Future work

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5. Study complexity depending on noise parameter ε.
6. Instead of function sequence f_u, encode everything in one function.

Continuous-time dynamical systems with noise

Definition

A Wiener-process is a stochastic process $(W_t)_{t \in \mathbb{R}^+}$ such that

1. $W_0 = 0$
2. $W_t - W_s$ is independent of W_u for all $0 \leq u \leq s$ and $t > s$
3. $W_t - W_s \sim \mathcal{N}(0, t-s)$

Definition

A stochastic ordinary differential equation (SDE) is an equation of the form

$$dX_t = f(X_t, t)dt + \sigma(X_t, t)dW_t.$$

The solution of a stochastic ODE is defined by Ito Integrals.

The Fokker-Planck equation

The probability densitities for the transition properties of the solution X_t to the SDE

$$dX_t = f(X_t, t)dt + \sigma(X_t, t)dW_t.$$

satisfy the Fokker-Planck equation

$$\frac{\partial p(x,t)}{\partial t} = -\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left[f_i(x,t)p(x,t)\right] + \sum_{i=1}^{N}\sum_{j=1}^{N} \frac{\partial^2}{\partial x_i\,\partial x_j}\left[D_{ij}(x,t)p(x,t)\right]$$

where

$$D = \frac{1}{2}\sigma\sigma^{\mathsf{T}}$$

Invariant density: $\frac{\partial p}{\partial t} = 0$.

Invariant measure for time-continuous noisy systems

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$$p(x) = \frac{1}{N} e^{-2F(x)\varepsilon^{-2}}$$

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Future work: Complexity for the general case.