McShane-Whitney extensions and the Hahn-Banach theorem

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Overview of this talk

- Lipschitz functions, constructively
- The McShane-Whitney extension theorem
- From Hahn-Banach to McShane-Whitney
- From McShane-Whitney to Hahn-Banach

Lipschitz functions, constructively

Lipschitz functions

$$\mathrm{Lip}(X, Y) := \bigcup_{\sigma \geq 0} \mathrm{Lip}(X, Y, \sigma),$$

$$\mathrm{Lip}(X, Y, \sigma) := \{f \in \mathbb{F}(X, Y) \mid \forall_{x,y \in X}(\rho(f(x), f(y)) \leq \sigma d(x, y))\}.$$

If $Y = \mathbb{R}$, we write $\mathrm{Lip}(X)$ and $\mathrm{Lip}(X, \sigma)$, respectively.

$$\mathrm{Lip}(X, Y) \subseteq C_u(X, Y)$$

If $f \in \mathrm{Lip}(X, Y)$, then f respects boundedness. If $\mathbb{N}$ is with discrete metric, then $\mathrm{id} : \mathbb{N} \to \mathbb{R} \in C_u(\mathbb{N}) \setminus \mathrm{Lip}(\mathbb{N})$ and $\mathrm{id}(\mathbb{N}) = \mathbb{N}$.

Met: the category of metric spaces with arrows between X, Y the set $\mathrm{Lip}(X, Y, 1)$.

Proposition (P, 2016)

Let X be a totally bounded metric space. If $f \in C_u(X)$ and $\epsilon > 0$, there are $\sigma > 0$ and $g^*, {}^*g \in \mathrm{Lip}(X, \sigma)$ s.t.

(*i*) $f - \epsilon \leq g^* \leq f \leq {}^*g \leq f + \epsilon$.
(*ii*) *For every* $e \in \mathrm{Lip}(X, \sigma)$, $e \leq f \Rightarrow e \leq g^*$.
(*iii*) *For every* $e \in \mathrm{Lip}(X, \sigma)$, $f \leq e \Rightarrow {}^*g \leq e$.

Corollary

If X is totally bounded, then $\mathrm{Lip}(X)$ is uniformly dense in $C_u(X)$.

Uniformly continuous functions "are" almost Lipschitz

Definition

A function $f : X \to Y$ is **almost Lipschitz**, if there is a *modulus of almost Lipschitz-continuity* $\sigma_f : \mathbb{R}^+ \to \mathbb{R}^+$ s.t.

$$\forall_{\epsilon>0}\big(f \in \mathrm{Lip}_\epsilon(X, Y, \sigma_f(\epsilon))\big),$$

$$f \in \mathrm{Lip}_\epsilon(X, Y, \sigma) :\Leftrightarrow \forall_{x,y\in X}\big(\rho(f(x), f(y)) \leq \sigma d(x, y) + \epsilon\big).$$

If f is almost Lipschitz, then f is uniformly continuous and respects boundedness.

Theorem (Vanderbei, 2017)

Let X, Y be normed spaces and let $C \subseteq X$ be convex. If $f : C \to Y$ is uniformly continuous, then f is almost Lipschitz.

Proof.

Constructive. □

The uniform limit of almost Lipschitz functions is almost Lipschitz.

$$\Lambda(f) := \{\sigma \geq 0 \mid \forall_{x,y \in X}(\rho(f(x), f(y)) \leq \sigma d(x,y))\},$$

$$M_0(f) := \{\sigma_{x,y}(f) \mid (x,y) \in X_0\},$$

$$X_0 := \{(x,y) \in X \times X \mid d(x,y) > 0\},$$

$$\sigma_{x,y}(f) := \frac{\rho(f(x), f(y))}{d(x,y)}.$$

Classically, if $f \in \mathrm{Lip}(X, Y)$ and $\exists \inf \Lambda(f)$, then $\exists \sup M_0(f)$, and $\sup M_0(f) = \inf \Lambda(f)$.

Proposition

Let $f \in \mathrm{Lip}(X, Y)$.
(i) If $\exists \sup M_0(f)$, *then* $\exists \inf \Lambda(f)$, *and*
$\inf \Lambda(f) = \min \Lambda(f) = \sup M_0(f)$.
(ii) If $\exists \inf \Lambda(f)$, *then* $\exists \mathrm{lub} M_0(f)$ *and* $\mathrm{lub} M_0(f) = \inf \Lambda(f)$.
(iii) If $\exists \mathrm{lub} M_0(f)$, *then* $\exists \inf \Lambda(f)$ *and* $\inf \Lambda(f) = \mathrm{lub} M_0(f)$.

$$L(f) := \sup M_0(f), \quad \text{the Lipschitz constant of } f,$$

$$L^*(f) := \mathrm{lub} M_0(f), \quad \text{the weak Lipschitz constant of } f.$$

Open problem: To find conditions on $X, Y, f \in \mathrm{Lip}(X, Y)$ such that $L(f)$ and/or $L^*(f)$ exist.

Lebesgue: If $f : (a, b) \to \mathbb{R}$ is Lipschitz, then f is differentiable almost everywhere.

Rademacher: Let $U \subseteq \mathbb{R}^n$ be open. If $f : U \to \mathbb{R}^m$ is Lipschitz, then f is almost everywhere differentiable.

Demuth (1969): In RUSS there is a Lipschitz function $f : [0, 1] \to \mathbb{R}$, which is nowhere differentiable.

The McShane-Whitney extension theorem

1. McShane-Whitney extension theorem (1934): A real-valued Lipschitz function defined on **any** subset A of a metric space X is extended to a Lipschitz function defined on X.

2. It has a highly ineffective proof with the use of Zorn's lemma, similar to the proof of the analytic Hahn-Banach theorem.

3. It also admits a proof based on an explicit definition of two such extension functions. This definition, which involves the notions of infimum and supremum of a non-empty bounded subset of $\mathbb{R}$, can be carried out constructively only if we restrict to certain subsets A of a metric space X.

4. To determine metric spaces X and Y such that a similar extension theorem for Y-valued Lipschitz functions defined on a subset A of X holds is a non-trivial problem under active current study in classical analysis.

Definition

Let $A \subseteq X$. We call (X, A) a **McShane-Whitney pair**, if for every $\sigma > 0$ and $g \in \mathrm{Lip}(A, \sigma)$ the functions $g^*, {}^*g : X \to \mathbb{R}$ are well-defined,

$$g^*(x) := \sup\{g(a) - \sigma d(x, a) \mid a \in A\},$$

$$^*g(x) := \inf\{g(a) + \sigma d(x, a) \mid a \in A\}.$$

Theorem (McShane-Whitney)

If (X, A) is an MW-pair and $g \in \mathrm{Lip}(A, \sigma)$, then:

(i) $g^*, {}^*g \in \mathrm{Lip}(X, \sigma)$.

(ii) ${g^*}_{|A} = ({}^*g)_{|A} = g$.

(iii) $\forall_{f \in \mathrm{Lip}(X,\sigma)}(f_{|A} = g \Rightarrow g^* \leq f \leq {}^*g)$.

(iv) The pair $(g^*, {}^*g)$ is the unique pair of functions satisfying (i)-(iii).

In the constructive proof the properties of lub and glb are used, hence we could have defined g^*, ${}^*g$ through lub and glb.

Proposition

The following pairs (X, A) are MW-pairs:

(i) A is totally bounded subset of X.

(ii) X is totally bounded and A is located.

(iii) X is locally compact (totally bounded) and A is bounded and located.

(iv) A is dense in X. In this case $g^* = {}^*g$.

Open problem: to completely determine the MW-pairs.

Proposition

Let (X, A) *be a MW-pair and* $g \in \mathrm{Lip}(A, \sigma)$*.*

(i) The set A *is located.*

(ii) If $\inf g$ *and* $\sup g$ *exist, then* $\inf {}^*g$, $\sup g^*$ *exist and*

$$\inf_{x\in X} {}^*g = \inf_{a\in A} g, \qquad \sup_{x\in X} g^* = \sup_{a\in A} g.$$

Proposition (step-invariance)

If $A \subseteq B \subseteq X$ *such that* $(X, A), (X, B)$*,* (B, A) *are MW-pairs and* $g \in \mathrm{Lip}(A, \sigma)$*, for some* $\sigma > 0$*, then*

$$g^{*X} = g^{*B*X}, \quad {}^{*X}g = {}^{*X*B}g.$$

Proposition

Let (X, A) *be a MW-pair and* $g \in \mathrm{Lip}(A)$ *such that* $L(g)$ *exists.*

(i) $g \in \mathrm{Lip}(A, L(g))$.

(ii) If f *is an* $L(g)$*-Lipschitz extension of* g*, then* $L(f)$ *exists and* $L(f) = L(g)$.

(iii) $L({}^*g), L(g^*)$ *exist and* $L({}^*g) = L(g) = L(g^*)$.

Proposition

Let (X, A) *be MW-pair,* $g_1 \in \mathrm{Lip}(A, \sigma_1), g_2 \in \mathrm{Lip}(A, \sigma_2)$ *and* $g \in \mathrm{Lip}(A, \sigma)$*, for some* $\sigma_1, \sigma_2, \sigma > 0$.

(i) $(g_1 + g_2)^* \leq g_1^* + g_2^*$ *and* ${}^*(g_1 + g_2) \geq {}^*g_1 + {}^*g_2$.

(ii) If $\lambda > 0$*, then* $(\lambda g)^* = \lambda g^*$ *and* ${}^*(\lambda g) = \lambda {}^*g$.

(iii) If $\lambda < 0$*, then* $(\lambda g)^* = \lambda {}^*g$ *and* ${}^*(\lambda g) = \lambda g^*$.

Proposition

Let $(X, ||.||)$ *be a normed space,* $C \subseteq X$ *convex,* (X, C) *a MW-pair, and* $g \in \mathrm{Lip}(C, \sigma)$*, for some* $\sigma > 0$*.*

(i) If g *is convex, then* ${}^{*}g$ *is convex.*

(ii) If g *is concave, then* g^{*} *is concave.*

Definition

(X, A) is a **locally** MW-pair, if for every bounded $B \subseteq A$ there is $B \subseteq A' \subseteq A$ such that (X, A') is a MW-pair.

Proposition (Bridges-Vîță)

Let Y be a located subset of a metric space X and T a totally bounded subset of X such that $T \between Y$. Then there is $S \subseteq X$ totally bounded such that $T \cap Y \subseteq S \subseteq Y$.

Proposition

(i) If X is locally totally bounded metric space and $A \subseteq X$ located, then (X, A) is a locally MW-pair.

(ii) If A is a locally totally bounded subset of X, then (X, A) is a locally MW-pair.

Hölder continuous functions of order α

If $\sigma \geq 0$ and $\alpha \in (0,1]$,

$$\mathrm{Höl}(X,Y,\sigma,\alpha) := \{f \in \mathbb{F}(X,Y) \mid \forall_{x,y \in X}(\rho(f(x),f(y)) \leq \sigma d(x,y)^{\alpha})\},$$

$$\mathrm{Höl}(X,Y,\alpha) := \bigcup_{\sigma \geq 0} \mathrm{Höl}(X,Y,\sigma).$$

If $Y = \mathbb{R}$, we write $\mathrm{Höl}(X,\sigma,\alpha)$ and $\mathrm{Höl}(X,\alpha)$.
If $g : A \to \mathbb{R} \in \mathrm{Höl}(A,\sigma,\alpha)$, then

$$g^*_\alpha(x) := \sup\{g(a) - \sigma d(x,a)^{\alpha} \mid a \in A\},$$

$$^*g_\alpha(x) := \inf\{g(a) + \sigma d(x,a)^{\alpha} \mid a \in A\}.$$

MW-pairs w.r.t. Hölder continuous functions and similarly shown MW-extension.

Definition

A **modulus of continuity** is a $\lambda : [0, +\infty) \to [0, +\infty)$ s.t.

(i) $\lambda(0) = 0$.

(ii) $\forall_{x,y\in[0,+\infty)}(\lambda(x + y) \leq \lambda(x) + \lambda(y))$.

(iii) It is strictly increasing i.e., $\forall_{s,t\in[0,+\infty)}(s < t \to \lambda(s) < \lambda(t))$.

(iv) It is uniformly continuous on every bounded subset of $[0, +\infty)$.

$$S(X, Y, \lambda) := \{f \in C_u(X, Y) \mid \forall_{x,y\in X}(\rho(f(x), f(y)) \leq \lambda(d(x, y))\},$$

$$S(X, \lambda) := \{f \in C_u(X) \mid \forall_{x,y\in X}(|f(x) - f(y)| \leq \lambda(d(x, y))\},$$

If $\lambda_1(t) = \sigma t$, $\lambda_2(t) = \sigma t^\alpha$, then

$$S(X, \lambda_1) = \mathrm{Lip}(X, \sigma),$$

$$S(X, \lambda_2) = \mathrm{H\ddot{o}l}(X, \sigma, \alpha).$$

Theorem (Bishop-Bridges)

Let (X, d) be a totally bounded metric space, $M > 0$ and λ a modulus of continuity. The set

$$S(\lambda, M) := \{f \in S(X, \lambda) \mid ||f||_\infty \leq M\}$$

is compact.

If $g : A \to \mathbb{R} \in S(A, \lambda)$,

$$g^*_\lambda(x) := \sup\{g(a) - \lambda(d(x, a)) \mid a \in A\},$$

$$^*g_\lambda(x) := \inf\{g(a) + \lambda(d(x, a)) \mid a \in A\}.$$

MW-pairs w.r.t. λ-continuous functions and similarly shown MW-extension.

From Hahn-Banach to McShane-Whitney

Proposition

Let $(X, ||.||)$ *be a normed space,* A *a non-trivial subspace of* X *such that* (X, A) *is an MW-pair, and let* $g \in \mathrm{Lip}(A, \sigma)$ *be linear.*

(i) $g^*(x_1 + x_2) \geq g^*(x_1) + g^*(x_2), \ \ {}^*g(x_1 + x_2) \leq {}^*g(x_1) + {}^*g(x_2)$.

(ii) If $\lambda > 0$*, then* $g^*(\lambda x) = \lambda g^*(x)$ *and* ${}^*g(\lambda x) = \lambda^* g(x)$.

(iii) If $\lambda < 0$*, then* $g^*(\lambda x) = \lambda^* g(x)$ *and* ${}^*g(\lambda x) = \lambda g^*(x)$.

I.e., ${}^*g$ is sublinear and g^* is superlinear.

Proposition

If $(X, ||.||)$ *is a normed space and* $x_0 \in X$ *such that* $||x_0|| > 0$*, there exists* $f \in \mathrm{Lip}(X)$ *such that* $f(x_0) = ||x_0||$ *and* $L(f) = 1$*.*

Proof.

If $\mathbb{I}x_0 := \{\lambda x_0 \mid \lambda \in [-1, 1]\}$, the function $g : \mathbb{I}x_0 \to \mathbb{R}$, defined by

$$g(\lambda x_0) = \lambda ||x_0||,$$

for every $\lambda \in [-1, 1]$, is in $\mathrm{Lip}(\mathbb{I}x_0)$ and $L(g) = 1$; if $\lambda, \mu \in [-1, 1]$, then $|g(\lambda x_0) - g(\mu x_0)| = \big|\lambda ||x_0|| - \mu ||x_0||\big| = |\lambda - \mu| ||x_0|| = ||(\lambda - \mu) x_0|| = ||\lambda x_0 - \mu x_0||$, and since

$$M_0(g) = \left\{ \sigma_{\lambda x_0, \mu x_0}(g) = \frac{|g(\lambda x_0) - g(\mu x_0)|}{||\lambda x_0 - \mu x_0||} = 1 \mid (\lambda, \mu) \in [-1, 1]_0 \right\},$$

we get that $L(g) = \sup M_0(g) = 1$. Since $\mathbb{I}x_0$ is inhabited and totally bounded, since it is compact, the extension ${}^*g$ of g is in $\mathrm{Lip}(X)$, and $L({}^*g) = L(g) = 1$. □

Theorem

Let $(X, ||.||)$ be a normed space and $x_0 \in X$ such that $||x_0|| > 0$. If $(X, \mathbb{R}x_0)$ is a MW-pair, there exist a sublinear Lipschitz function f on X such that $f(x_0) = ||x_0||$ and $L(f) = 1$, and a superlinear Lipschitz function h on X such that $h(x_0) = ||x_0||$ and $L(h) = 1$.

Proof.

As in the previous proof the function $g : \mathbb{R}x_0 \to \mathbb{R}$, defined by $g(\lambda x_0) = \lambda||x_0||$, for every $\lambda \in \mathbb{R}$, is in $\mathrm{Lip}(\mathbb{R}x_0)$ and $L(g) = 1$. Since $(X, \mathbb{R}x_0)$ is a MW-pair, the extension $^*g$ of g is a Lipschitz function, and $L(^*g) = L(g) = 1$. Since g is linear, we get that $^*g$ is sublinear. Similarly, the extension g^* of g is a Lipschitz function, and $L(g^*) = L(g) = 1$. Since g is linear, we get that g^* is superlinear. □

Open problem: To find conditions on $(X, ||.||)$ s.t. $(X, \mathbb{R}x_0)$ is a MW-pair, if $||x_0|| > 0$. A similar attitude is taken by Ishihara in his constructive proof of the Hahn-Banach theorem, where the property of Gâteaux differentiability of the norm is added, and X is uniformly convex and complete normed space.

From McShane-Whitney to Hahn-Banach

Analytic Hahn-Banach theorem for seminorms (Zorn)

Let X be a vector space and $p : X \to \mathbb{R}$ a *seminorm* on X i.e.,

(i) $p(x) \geq 0$,

(ii) $p(x + y) \leq p(x) + p(y)$,

(iii) $p(\lambda x) = |\lambda| p(x)$,

and let A subspace of X and $g : A \to \mathbb{R}$ linear such that

$$\forall_{a \in A} \big(g(a) \leq p(a) \big).$$

Then there is linear $f : X \to \mathbb{R}$ that extends g and

$$\forall_{x \in X} \big(f(x) \leq p(x) \big).$$

In its general formulation (iii) is replaced by

(iii') $p(\lambda x) = \lambda p(x)$, for $\lambda \geq 0$ [(i), (ii), (iii'): *sublinear functional*].

p-Lipschitz functions

If p is a seminorm on X, we define

$$p\text{--}\mathrm{Lip}(X) := \bigcup_{\sigma \geq 0} p\text{--}\mathrm{Lip}(X, \sigma),$$

$$p\text{--}\mathrm{Lip}(X, \sigma) := \{f \in \mathbb{F}(X, Y) \mid \forall_{x,y \in X}(|f(x) - f(y)| \leq \sigma p(x - y)\}.$$

Remark

Let A be a subspace of the normed space X and $g : A \to \mathbb{R}$.

(i) If g is subadditive and $g(a) \leq p(a)$, for every $a \in A$, then for every $a, b \in A$

$$|g(a) - g(b)| \leq p(a - b)$$

i.e., $g \in p\text{--}\mathrm{Lip}(A, 1)$.

(ii) If g is superadditive and $g(a) \geq p(a)$, for every $a \in A$, then for every $a, b \in A$

$$|g(a) - g(b)| \geq p(a - b).$$

Definition

Let A be a subset of the normed space X and p a seminorm on X. We call (X, A) a **p-McShane-Whitney pair**, if for every $\sigma > 0$ and every $g \in p{-}\mathrm{Lip}(A, \sigma)$ the functions $g^*, {}^*g : X \to \mathbb{R}$ are well-defined,

$$g^*(x) := \sup\{g(a) - \sigma p(x - a) \mid a \in A\},$$

$$^*g(x) := \inf\{g(a) + \sigma p(x - a) \mid a \in A\}.$$

Theorem (McShane-Whitney for p-Lipschitz functions)

If (X, A) is a p MW-pair and $g \in p{-}\mathrm{Lip}(A, \sigma)$, then:

(i) $g^*, {}^*g \in p{-}\mathrm{Lip}(X, \sigma)$.

(ii) ${g^*}_{|A} = ({}^*g)_{|A} = g$.

(iii) $\forall_{f \in p-\mathrm{Lip}(X,\sigma)}(f_{|A} = g \Rightarrow g^* \leq f \leq {}^*g)$.

(iv) The pair $(g^*, {}^*g)$ *is the unique pair of functions satisfying (i)-(iii).*

MW-version of the analytic Hahn-Banach for seminorms

Theorem

Let p be a seminorm on X, (X, A) a p-MW-pair, A a subspace of X, and $g : A \to \mathbb{R}$ linear.

(i) If $\forall_{a \in A}(g(a) \leq p(a))$, there is a superlinear extension g^* of g such that $g^* \in p{-}\mathrm{Lip}(X, 1)$ and $\forall_{x \in X}(g^*(x) \leq p(x))$.

(ii) If $\forall_{a \in A}(g(a) \geq p(a))$, there is a sublinear extension ${}^*g$ of g such that ${}^*g \in p{-}\mathrm{Lip}(X, 1)$ and $\forall_{x \in X}({}^*g(x) \geq p(x))$.

Proof.

(i) By the previous remark $g \in p{-}\mathrm{Lip}(A, 1)$, and we use the McShane-Whitney extension theorem for p-Lipschitz functions.
(ii) Similarly. □

W.r.t. analytic HB for seminorms, we lost linearity (in (i) we have only superlinearity), but we have that $g^* \in p{-}(X, 1)$.

Of course, we didn't use Zorn's lemma.

We hope to find good applications of this in convex analysis.

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