Strong Completeness & the Finite Model Property for Bi-Intuitionistic Stable Tense Logics (**BiSKt**s)

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2nd Workshop on MLA @ Kanazawa, 8th March 2018
Joint Work w/ John G. Stell (University of Leeds)

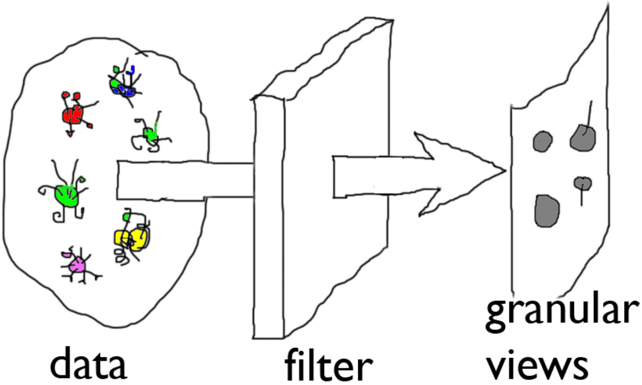
Outline

1. Mathematical Morphology: From Sets to Graphs
2. Syntax and Semantics of **BiSKt**
3. Hilbert System of **BiSKt**: Strong Completeness & FMP

Outline

What is Mathematical Morphology?

- A technique for image processing.



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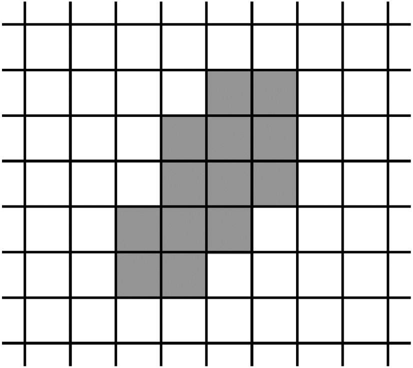
Set-based Mathematical Morphology

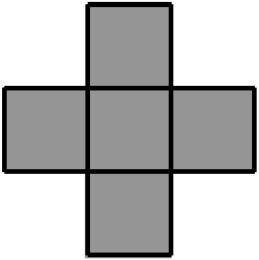
- Let $U = \mathbb{Z}^2$, a grid of pixels.
- A subset X of $\mathbb{Z}^2$: a binary image (black and white).



X

Structuring Element

- is also called a probe or lens.
- induces a binary relation R on $X = \mathbb{Z}^2$.



Example of Structuring Element

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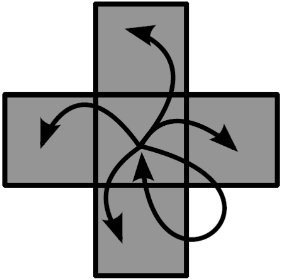
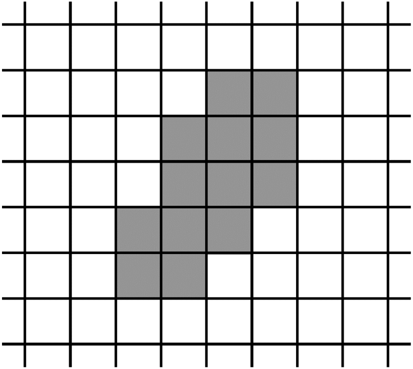


R induced by the Structuring Element

Dilation by R

Let U be a set, $R \subseteq U^2$ and $X \subseteq U$.

$$X \oplus R \quad := \quad \{\, u \in U \mid \exists\, v.(vRu \text{ and } v \in X)\,\}$$



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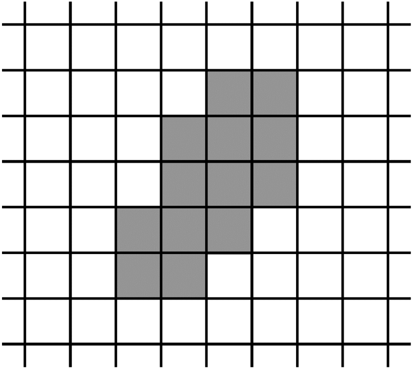
$$:=\; \blacklozenge X$$

where "$\blacklozenge$" is a backward looking (past tense) operator.



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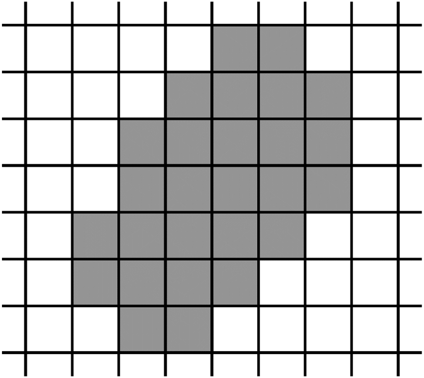
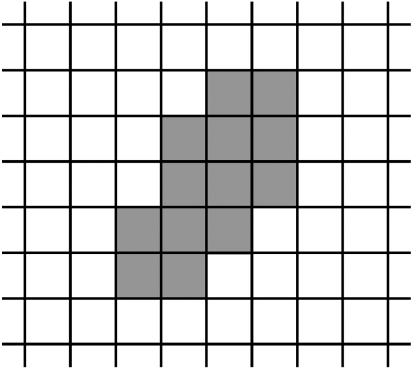


$X \oplus R$

Erosion by R

Let U be a set, $R \subseteq U^2$ and $X \subseteq U$.

$$R \ominus X \quad := \quad \{\, u \in U \mid \forall\, v.(uRv \text{ implies } v \in X) \,\}$$



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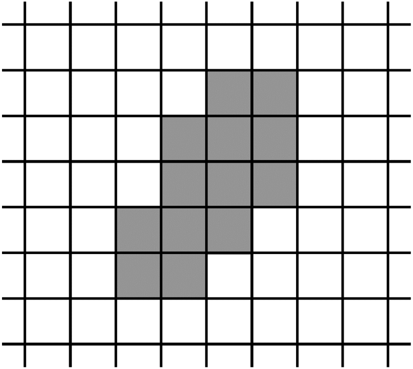

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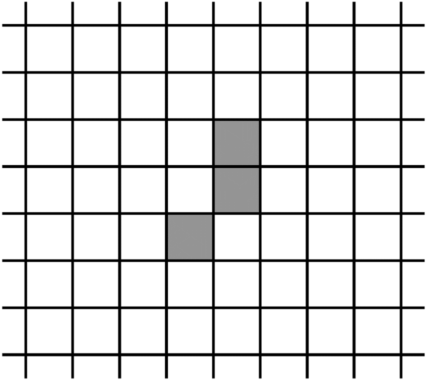
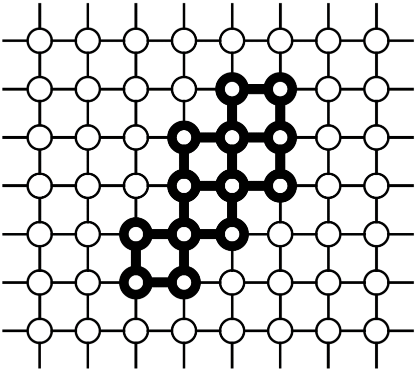


$R \ominus X$

Mathematical Morphology for Graphs

- A graph consists of nodes and edges.
- A subgraph is naturally defined.



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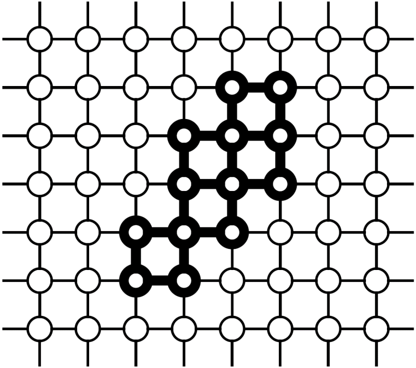

What is a relation of a graph?

Stell (2015)'s Approach: Hypergraphs

Let us "mix" nodes and edges into one domain!

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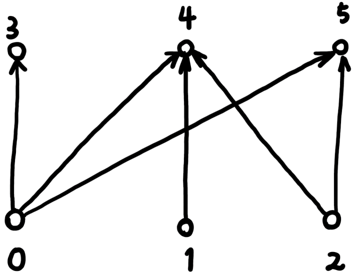
(U, H) is a hypergraph if:

- (U, H) is a preorder (reflexive and transitive);
- If xHy and yHz then $x = y$ or $y = z$ for all $x, y, z \in U$.



How to Recover Edges and Nodes

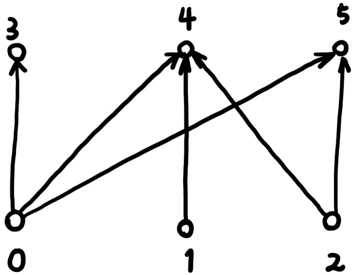
Let (U, H) be a hypergraph.

- $u \in U$ is a node if u is an H-maximal element, i.e.,

$$\forall v.(uHv \text{ implies } u = v)$$

- $u \in U$ is an edge if u has a proper H-successor, i.e.,

$$\exists v.(uHv \text{ and } u \neq v)$$



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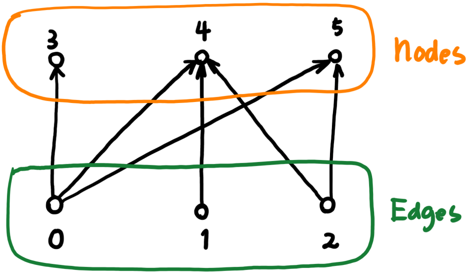
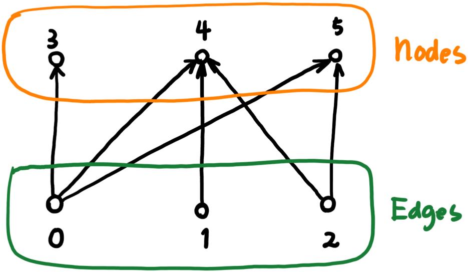


Graphs From Viewpoints of Hypergraphs

Let (U, H) be a hypergraph.

An edge v and a node u is incident if vHu holds

(U, H) is a graph if all edges are incident w/ one or two nodes.



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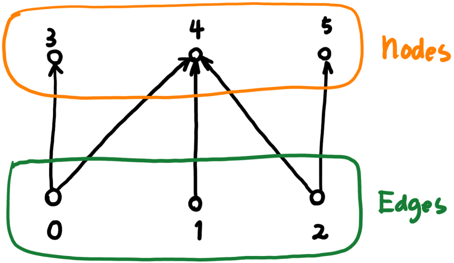

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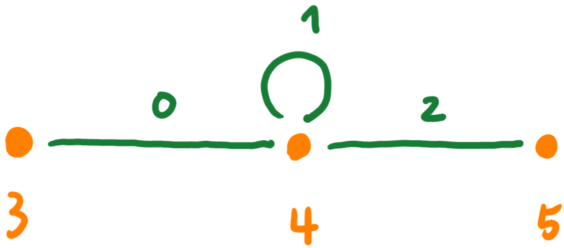
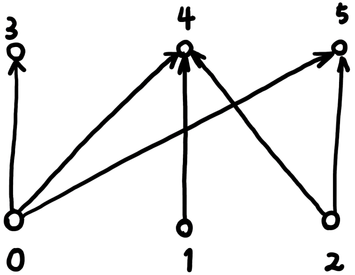


Subgraph of a Hypergraph

Let (U, H) be a hypergraph.

- $X \subseteq U$ is a subgraph of (U, H) if X is H-closed, i.e., X is closed under taking H-successors.



Binary Relation on a Hypergraph

Over Sets: it is known that

$$\{ \text{ all binary rel.s on } U \} \cong \{ \text{ all } \cup\text{-preserving maps on } \mathcal{P}(U) \}$$

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Over (Hyper) Graphs: Let (U, H) be a preorder.

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where $\mathcal{P}^{\uparrow}(U)$ is the set of all H-closed sets.

- $R \subseteq U^2$ is stable if:

$$H; R; H \subseteq R$$

where ; is a relational composition.

What is Bi-intuitionistic Stable Tense Logic?

The logic of all preorders (U, H) with a stable relation R!

- **BiSKt** = Bi-intuitionistic logic + $\blacklozenge$ + $\Box$.

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- Why "bi"-intuitionistic logic (Rauszer 1974)?

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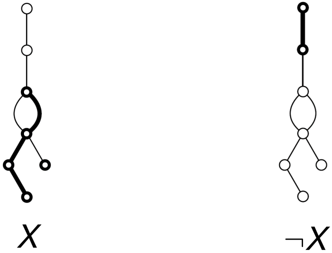

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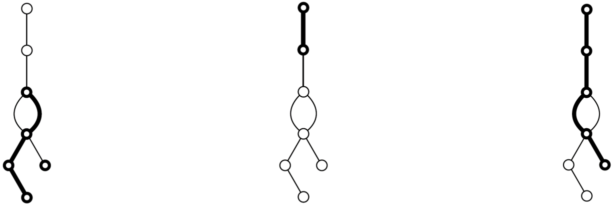

X $\qquad$ $\neg X$ $\qquad$ $\lrcorner X$

Background & Contribution of This Talk

For **BiSKt**, Stell, Schmidt & Rydeheard (2016) provided

- Labelled tableau calculus of **BiSKt**
- Semantic completeness and decidability of it
- Frame definability results

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For **BiSKt**, this talk provides:

1. Hilbert-style axiomatization of **BiSKt**.
2. Strong completeness results of extensions of **BiSKt**
3. FMP via filtration for extensions of **BiSKt**

Outline

Syntax of **BiSKt**

Let Prop be a countable set of propositional variables.

$$\varphi ::= \top \mid \bot \mid p \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \varphi \to \varphi \mid \varphi \prec \varphi \mid \blacklozenge \varphi \mid \Box \varphi \quad (p \in \text{Prop}).$$

- $\neg \varphi := \varphi \to \bot$.
- $\varphi \prec \psi$ is read as: "φ excludes ψ": Coimplication
- $\lrcorner\, \psi := \top \prec \psi$: Conegation

H-Frame and H-Model

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An H-model M consists of:

an H-frame (U, H, R) and a valuation V:

$$V : \text{Prop} \to \mathcal{P}^{\uparrow}(U) = \{\, \text{all } H\text{-closed sets on } U \,\}.$$

Kripke Semantics of **BiSKt**

Let $M = (U, H, R, V)$ be an H-model. For a state $u \in U$ and a formula φ, the satisfaction $M, u \models \varphi$ is defined as:

$$M, u \models \varphi \to \psi \quad \text{iff} \quad \forall v \in U\,((uHv \,\&\, M, v \models \varphi) \Rightarrow M, v \models \psi),$$

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Define $[\![\varphi]\!] := \{\, u \in U \mid M, u \models \varphi \,\}$.

$[\![\varphi]\!]$ is H-closed.

$[\![\varphi]\!]$ is a subgraph when (U, H, R) is a hypergraph.

Frame Definability (Stell, Schmidt & Rydeheard 2016)

Define the frame validity $F \models \varphi$ as in ordinary modal logic.

Let $F = (U, H, R)$ be an H-frame. TFAE:

1. $R^m \subseteq R^n$.
2. $F \models \blacklozenge^m p \to \blacklozenge^n p$.

where $R^k := \underbrace{R; \cdots ; R}_{k}$ ($R^0 := H$) and $\blacklozenge^k := \underbrace{\blacklozenge \cdots \blacklozenge}_{k}$.

Note: this talk simplifies their more general results.

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3. Hilbert System of **BiSKt**: Strong Completeness & FMP

Residuation or Adjunction

Let $\models \varphi$ mean that $F \models \varphi$ for all H-frames F. We have:

$$\models (p \wedge q) \to r \iff \models p \to (q \to r).$$

For coimplication,

$$\models (p \prec q) \to r \iff \models p \to (q \vee r).$$

For $\blacklozenge$ and $\Box$,

$$\models \blacklozenge p \to q \iff \models p \to \Box q.$$

Hilbert System of **BiSKt**

To Hilbert system of **Int** (w/ uniform substitution), we add:

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To Hilbert system of **Int** (w/ uniform substitution), we add:

For $\blacklozenge$ and $\square$,

(A12) $p \to \square \blacklozenge p$

(A13) $\blacklozenge \square p \to p$

(Mon$\blacklozenge$) From $\varphi \to \psi$, infer $\blacklozenge \varphi \to \blacklozenge \psi$.

(Mon$\square$) From $\varphi \to \psi$, infer $\square\varphi \to \square\psi$.

Hilbert System of **BiSKt**

To Hilbert system of **Int** (w/ uniform substitution), we add:
For coimplicaiton,

(A10) $p \to (q \vee (p \prec q))$

(A11) $((q \vee r) \prec q) \to r$

(Mon$\prec$) From $\delta_1 \to \delta_2$, infer $(\delta_1 \prec \psi) \to (\delta_2 \prec \psi)$,

For $\blacklozenge$ and $\Box$,

(A12) $p \to \Box \blacklozenge p$

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For $\blacklozenge$ and $\Box$,

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(Mon$\blacklozenge$) From $\varphi \to \psi$, infer $\blacklozenge \varphi \to \blacklozenge \psi$.

(Mon$\Box$) From $\varphi \to \psi$, infer $\Box\varphi \to \Box\psi$.

For **BiInt**, it's much simpler than Rauszer (1974)'s system.

Strong Completeness for Extensions of **BiSKt**

Let Σ be a set of axioms of the form:

$$\blacklozenge^m p \to \blacklozenge^n p$$

Then **BiSKt** + Σ is strongly complete for the class of H-frames defined by Σ.

BiSKt + Σ is strongly complete for a class $\mathbb{F}$ of H-frames if every consistent set Γ in the extension is satisfiable in $\mathbb{F}$.

FMP for Extensions of **BiSKt**

Let Σ be a finite set of axioms of the form:

$$\blacklozenge p \to \blacklozenge^n p \text{ or } p \to \blacklozenge^n p$$

Then **BiSKt** $+ \Sigma$ enjoys the finite model property for the class of H-frames defined by Σ, so it is decidable.

BiSKt $+ \Sigma$ enjoys FMP for a class $\mathbb{F}$ of H-frames if every consistent formula φ in the extension is satisfiable in a finite frame in $\mathbb{F}$.

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BiSKt + Σ enjoys FMP for a class $\mathbb{F}$ of *H*-frames if every consistent formula φ in the extension is satisfiable in a finite frame in $\mathbb{F}$.

($\because$) By filtration technique by Hasimoto (2001) for intuitionistic modal logic.

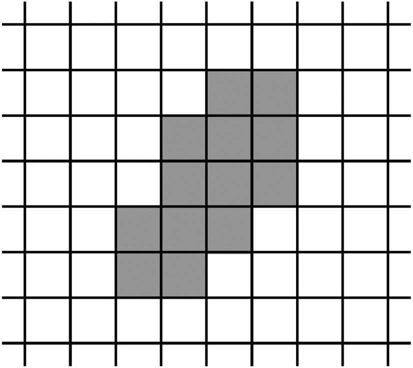
Further Directions

- Strong Comp. & FMP resutls can be extended even if we mix $\Diamond := \lrcorner\Box\neg$ ("for some future") with $\blacklozenge$.
- More applications (in my last visit to Leeds):
  - Formalize spatial relationship over discrete space
  - Discrete ver. RCC8. Universal modalities are needed.
- (Done) Sequent calculus w/ the analytic cut rule
  - (cf.) Kowalski & Ono (2016): Craig Interpolation for **BiInt**
  - Joint work w/ Hiroakira Ono
- More general results on completeness & FMP
  - (cf.) Wolter (1998): On Logics with Coimplication (Modal Expansion of **BiInt**)

Opening of X by R

Let U be a set, $R \subseteq U^2$ and $X \subseteq U$.

$$(R \ominus X) \oplus R \quad (= \blacklozenge \Box X)$$

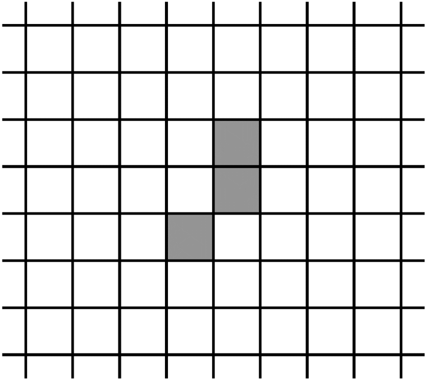


X

Opening of X by R

Let U be a set, $R \subseteq U^2$ and $X \subseteq U$.

$$(R \ominus X) \oplus R \quad (= \blacklozenge \Box X)$$



$R \ominus X$

Opening of X by R

Let U be a set, $R \subseteq U^2$ and $X \subseteq U$.

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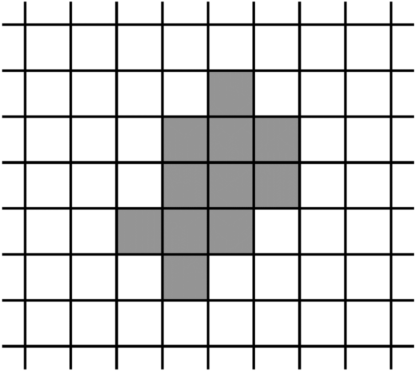

$(R \ominus X) \oplus R$

Application to Mathematical Morphology

Recall that "$\blacklozenge\,\Box$" corresponds to "opening by R".

$$\vdash_{\textbf{BiSKt}} \overbrace{\blacklozenge\,\Box}\,\varphi \to \varphi.$$

$$\vdash_{\textbf{BiSKt}} \overbrace{\blacklozenge\,\Box}\,\varphi \leftrightarrow \overbrace{\blacklozenge\,\Box}\;\overbrace{\blacklozenge\,\Box}\,\varphi.$$

Once we take the opening of X by R,

it becomes a fixed point of the opening by R!