MacNeille completion and Buchholz' Omega rule

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Summary of this talk

Buchholz' Ω-rule (1981)

$$\frac{\{\ \Delta \Rightarrow \Pi^{\theta}\ \}_{\Delta \Rightarrow^{\mathbf{LI}}_{Y} \varphi^{\theta}(Y)}}{\forall X.\varphi(X) \Rightarrow \Pi}$$

where Δ is 1st order and Π is 2nd order,

is similar to

a characteristic property of MacNeille completion $\mathbf{A} \subseteq \overline{\mathbf{A}}$:

$$\frac{\{a \leq y\}_{a \leq x}}{x \leq y}$$

where $a \in \mathbf{A}$ and $x, y \in \overline{\mathbf{A}}$.

Syntactic cut elimination

1. Ordinal assignment
2. Ω-rule technique (Buchholz, Aehlig, Mints, Akiyoshi, ...). Works only for fragments of higher order logic/arithmetic (so far).

Semantic cut elimination

1. Semi-valuation (Schütte, Takahashi, Prawitz). 3-valued semantics (Girard 74) = Kleene's semantics. Employs reductio ad absurdum and WKL. Destroys the proof structure.
2. MacNeille completion and reducibility candidates (Maehara 91, Okada 96, after Girard 71). Fully constructive. Extends to strong normalization.

Cut elimination proofs for higher order logics/arithmetic

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2. Ω-rule technique (Buchholz, Aehlig, Mints, Akiyoshi, ...). Works only for fragments of higher order logic/arithmetic (so far).

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2. MacNeille completion and reducibility candidates (Maehara 91, Okada 96, after Girard 71). Fully constructive. Extends to strong normalization.

What is the relationship? (Mints' question)

Target system	Fragments	Full higher-order logics
Algebraic proof	???	MacNeille completion $+$ reducibility candidates
Syntactic proof	Ω-rule	Takeuti's Conjecture

In this talk we fill in the ??? slot by introducing the concept of Ω-valuation. The target systems are parameter-free 2nd order intuitionistic logics.

Target system	Fragments	Full higher-order logics
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In this talk we fill in the ??? slot by introducing the concept of Ω-valuation. The target systems are parameter-free 2nd order intuitionistic logics.

Notice: It is mostly a reworking of known results (especially those of Klaus Aehlig). Our purpose is just to provide an algebraic perspective on them.

Outline

- MacNeille completion
- Parameter-free 2nd order intuitionistic logics
- Ω-rule technique (syntactic)
- Ω-valuation technique (semantic)
- For the lambda calculus audience
- For the nonclassical logics audience

MacNeille completion

MacNeille completion

$\mathbf{A}$: a lattice.
A completion of $\mathbf{A}$ is an embedding $e : \mathbf{A} \longrightarrow \mathbf{B}$ into a complete lattice $\mathbf{B}$ (we often assume $\mathbf{A} \subseteq \mathbf{B}$).

Examples:

- $\mathbb{Q} \subseteq \mathbb{R} \cup \{\pm\infty\}$
- $e : \mathbf{A} \longrightarrow \wp(\mathsf{uf}(\mathbf{A}))$ ($\mathbf{A}$: Boolean algebra, uf $=$ ultrafilters).

$\mathbf{A} \subseteq \mathbf{B}$ is a MacNeille completion if for any $x \in \mathbf{B}$,

$$x \;=\; \bigwedge\{a \in \mathbf{A} : x \leq_{\mathbf{B}} a\} \;=\; \bigvee\{a \in \mathbf{A} : a \leq_{\mathbf{B}} x\}.$$

Theorem (Banachewski 56, Schmidt 56)

Every lattice $\mathbf{A}$ has a unique MacNeille completion $\overline{\mathbf{A}}$.
MacNeille completion is regular, i.e., preserves $\bigwedge$ and $\bigvee$ that already exist in $\mathbf{A}$.

(Recap) $\mathbf{A} \subseteq \mathbf{B}$ is a MacNeille completion if for any $x \in \mathbf{B}$,

$$x \;=\; \bigwedge\{a \in \mathbf{A} : a \leq_{\mathbf{B}} x\} \;=\; \bigvee\{a \in \mathbf{A} : x \leq_{\mathbf{B}} a\}.$$

- $\mathbb{Q} \subseteq \mathbb{R} \cup \{\pm\infty\}$ is MacNeille, since

$$x = \inf\{a \in \mathbb{Q} : x \leq a\} = \sup\{a \in \mathbb{Q} : a \leq x\}$$

  for any $x \in \mathbb{R}$. It is regular, e.g.,

$$0 = \lim_{n\to\infty} \frac{1}{n} \text{ (in } \mathbb{Q}) = \lim_{n\to\infty} \frac{1}{n} \text{ (in } \mathbb{R}).$$

- $e : \mathbf{A} \longrightarrow \wp(\mathrm{uf}(\mathbf{A}))$ is not regular, hence not MacNeille (actually a canonical extension).

MacNeille completion: its limitation

$\mathcal{DL}$: the class of distributive lattices.
$\mathcal{HA}$: the class of Heyting algebras.
$\mathcal{BA}$: the class of Boolean algebras.

Theorem

- $\mathcal{DL}$ is not closed under MacNeille (Funayama 44).
- $\mathcal{HA}$ and $\mathcal{BA}$ are closed under MacNeille completions.
- These are the only nontrivial subvarieties of $\mathcal{HA}$ closed under MacNeille (Harding-Bezhanishvili 04).

Conservative extension by MacNeille completion does not work for proper intermediate logics.

Fact

A completion $\mathbf{A} \subseteq \mathbf{B}$ is MacNeille iff the inferences below are valid:

$$\frac{\{a \leq y\}_{a \leq x}}{x \leq y} \qquad \frac{\{x \leq a\}_{y \leq a}}{x \leq y}$$

where x, y range over $\mathbf{B}$ and a over $\mathbf{A}$.

"If $a \leq x$ implies $a \leq y$ for any $a \in \mathbf{A}$, then $x \leq y$."

This looks similar to the Ω-rule.

Parameter-free 2nd order intuitionistic logics

$\mathbf{ID}_0 := \mathbf{PA}$.

$\mathbf{ID}_1$: Let $\varphi(X, x)$ be a formula of $\mathbf{PA}(X)$ in which X occurs positively and $FV(\varphi) \subseteq \{X, x\}$.

It can be seen as a monotone function

$$\varphi(Y) := \{n \in \mathbb{N} : \varphi(Y, n) \text{ holds}\} : \wp(\mathbb{N}) \longrightarrow \wp(\mathbb{N}).$$

For each such φ, add to $\mathbf{PA}$ a new constant $\mathbf{I}_\varphi$ and axioms

$$\varphi(\mathbf{I}_\varphi) \subseteq \mathbf{I}_\varphi, \qquad \varphi(T) \subseteq T \Rightarrow \mathbf{I}_\varphi \subseteq T.$$

for every $T = \lambda x.\psi(x)$. This defines the theory $\mathbf{ID}_1$.

$\mathbf{ID}_{n+1} := \mathbf{ID}_n +$ least fixpoints definable in $\mathbf{ID}_n$

$\vdots$

$\mathbf{ID}_{<\omega} := \bigcup_n \mathbf{ID}_n$.

Tm: the set of terms
$X, Y, Z, \ldots$: 2nd order variables
Fm : the formulas of 1st-order intuitionistic logic

$$\varphi, \psi ::= p(\bar{t}) \mid t \in X \mid \bot \mid \varphi \wedge \psi \mid \varphi \vee \psi \mid \forall x.\varphi \mid \exists x.\varphi$$

FM_{-1} := Fm.
FM_{n+1}:

$$\varphi_{n+1} ::= p(\bar{t}) \mid t \in X \mid \cdots \mid \forall X.\varphi_n \mid \exists X.\varphi_n$$

where $\varphi_n \in \mathsf{FM}_n$ doesn't contain 2nd order variables except X.

(Recap) FM_{n+1}:

$$\varphi_{n+1} ::= \cdots \mid \forall X.\varphi_n \mid \exists X.\varphi_n$$

where $\varphi_n \in \mathsf{FM}_n$ doesn't contain 2nd order variables except X.

Examples (over $\mathcal{L}_{PA}$)

$\mathbf{N}(t) := \forall X.[\forall x(x \in X \to x{+}1 \in X) \wedge 0 \in X \to t \in X] \in \mathsf{FM}_0$

Any arithmetical formula φ translates to $\varphi^{\mathbf{N}} \in \mathsf{FM}_0$.
If $\varphi(X, x)$ is an arithmetical formula,

$\mathbf{I}_\varphi(t) := \forall X.[\forall x(\varphi^{\mathbf{N}}(X, x) \to x \in X) \to t \in X] \quad \in \mathsf{FM}_1$

Any formula φ of $\mathbf{ID}_1$ translates to $\varphi^{\mathbf{I}} \in \mathsf{FM}_1$.

FM: the set of all 2nd-order formulas.
$\mathbf{G^1LI}$: sequent calculus for 2nd order intuitionistic logic with full comprehension

$$\frac{\varphi(\lambda x.\psi), \Gamma \Rightarrow \Pi}{\forall X.\varphi(X), \Gamma \Rightarrow \Pi} \qquad \frac{\Gamma \Rightarrow_Y \varphi(Y)}{\Gamma \vdash \forall X.\varphi(X)}$$

where

- $\Gamma \Rightarrow_Y \varphi(Y)$ means $Y \notin FV(\Gamma)$ (eigenvariable).
- $\varphi(\lambda x.\psi)$ obtained by replacing $t \in X \mapsto \psi(t)$.

Theorem (Takeuti 53)

If $\mathbf{Z}_2 \vdash \varphi$, then $\mathbf{G^1LC} \vdash \Gamma_0 \Rightarrow \varphi^{\mathbf{N}}$ for some universal Γ_0.
Cut elimination for $\mathbf{G^1LC}$ implies 1-consistency of $\mathbf{Z}_2$, i.e., provable Σ^0_1-sentences are true.

$\mathbf{LI}$: sequent calculus for the 1st order intuitionistic logic.
$\mathbf{G^1LI}_{-1} := \mathbf{LI}$.
$\mathbf{G^1LI}_{n+1}$: sequent calculus $\mathbf{G^1LI}$ restricted to FM_{n+1}.

Theorem

If $\mathbf{ID}_n \vdash \varphi$ $(\in \Pi^0_2)$, then $\mathbf{G^1LI}_n \vdash \Gamma_0 \Rightarrow \varphi^{\mathbf{I}}$.
Cut elimination for $\mathbf{G^1LI}_n$ implies 1-consistency of $\mathbf{ID}_n$.

$\mathbf{LI}$: sequent calculus for the 1st order intuitionistic logic.
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Theorem

If $\mathbf{ID}_n \vdash \varphi$ ($\in \Pi^0_2$), then $\mathbf{G}^1\mathbf{LI}_n \vdash \Gamma_0 \Rightarrow \varphi^{\mathbf{I}}$.
Cut elimination for $\mathbf{G}^1\mathbf{LI}_n$ implies 1-consistency of $\mathbf{ID}_n$.

We are now interested in proving cut elimination for $\mathbf{G}^1\mathbf{LI}_n$ globally in $\mathbf{ID}_{n+1}$ and locally in $\mathbf{ID}_n$, as the latter will imply

$$1\mathsf{CON}(\mathbf{ID}_n) \leftrightarrow \mathsf{CE}(\mathbf{G}^1\mathbf{LI}_n)$$

in a suitably weak metatheory (eg., $\mathbf{PRA}$).

Ω-rule

Ω-rule: the motivation

Cut elimination for 2nd order logics is tricky, since the reduction step

$$\dfrac{\dfrac{\Gamma \Rightarrow_Y \varphi(Y)}{\Gamma \vdash \forall X.\varphi(X)} \quad \dfrac{\varphi(\lambda x.\psi), \Gamma \Rightarrow \Pi}{\forall X.\varphi(X), \Gamma \Rightarrow \Pi}}{\Gamma \Rightarrow \Pi}\ (CUT)$$

$$\Downarrow$$

$$\dfrac{\Gamma \Rightarrow \varphi(\lambda x.\psi) \quad \varphi(\lambda x.\psi), \Gamma \Rightarrow \Pi}{\Gamma \Rightarrow \Pi}\ (CUT)$$

may yield a BIGGER cut formula.
Ω-rule (Buchholz 81, Buchholz-Schütte 88, Buchholz 01, Aehlig 04, Akiyoshi-Mints 16, ...) is a way to resolve this difficulty.

The (simplified) Ω-rule for $\mathbf{G}^1\mathbf{LI}_0$:

$$\frac{\{\ \Delta \Rightarrow \Pi^\theta\ \}_{\Delta \Rightarrow^{\mathbf{LI}}_Y \varphi^\theta(Y)}}{\forall X.\varphi(X) \Rightarrow \Pi}$$

where θ is any substitution for 1st order free variables and $\Delta \Rightarrow^{\mathbf{LI}}_Y \varphi^\theta(Y)$ means

- $Y \notin \mathsf{FV}(\Delta)$,
- $\Delta \subseteq \mathsf{Fm}$ (1st order formulas),
- $\mathbf{LI} \vdash \Delta \Rightarrow \varphi^\theta(Y)$.

"If $\Delta \Rightarrow^{\mathbf{LI}}_Y \varphi^\theta(Y)$ implies $\Delta \Rightarrow \Pi^\theta$ for any θ and $\Delta \subseteq \mathsf{Fm}$, then $\forall X.\varphi(X) \Rightarrow \Pi$."

Embedding: We have:

$$\frac{\{\ \Delta \Rightarrow \varphi^{\theta}(\lambda x.\psi)\ \}_{\Delta \Rightarrow^{\mathbf{LI}}_{Y} \varphi^{\theta}(Y)}}{\forall X.\varphi(X) \Rightarrow \varphi(\lambda x.\psi)}$$

Hence $\forall X$-left can be simulated by Ω.

Collapsing: Consider

$$\cfrac{\cfrac{\Gamma \Rightarrow_{Y} \varphi(Y)}{\Gamma \Rightarrow \forall X.\varphi(X)} \qquad \cfrac{\{\ \Delta \Rightarrow \Pi^{\theta}\ \}_{\Delta \Rightarrow^{\mathbf{LI}}_{Y} \varphi^{\theta}(Y)}}{\forall X.\varphi(X) \Rightarrow \Pi}}{\Gamma \Rightarrow \Pi}\ (CUT)$$

If $\Gamma \Rightarrow^{\mathbf{LI}}_{Y} \varphi(Y)$ holds, then $\Gamma \Rightarrow \Pi$ is one of the premises (with $\theta = \mathrm{id}$). Hence the (CUT) can be eliminated.

Syntactic cut elimination for $\mathbf{G^1LI_0}$:

1. Introduce a new proof system based on the Ω-rule by inductive definition.
2. Show that $\mathbf{G^1LI_0}$ embeds into the new proof system.
3. Apply a syntactic cut elimination procedure.

It works for derivations of 1st order sequents.

Theorem

$\mathbf{ID}_1$ proves that $\mathbf{G^1LI_0}$ is a conservative extension of $\mathbf{LI}$.
$\mathbf{ID}_{n+1}$ proves that $\mathbf{G^1LI}_n$ is a conservative extension of $\mathbf{LI}$.

It can be extended to all derivations (Akiyoshi-Mints 16).

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It can be extended to all derivations (Akiyoshi-Mints 16).
So the Ω-rule works, but is it logically valid?

Ω-valuation

Warm-up: conservative extension by MacNeille completion

Let us first give an algebraic proof to

Fact

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(Proof)

Let $\mathbf{L} :=$ Fm$/\sim$ be the Lindenbaum algebra for $\mathbf{LI}$.

Let $\overline{\mathbf{L}}$ be the MacNeille completion of $\mathbf{L}$.

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Let $\mathbf{L} :=$ Fm$/\!\sim$ be the Lindenbaum algebra for $\mathbf{LI}$.
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The canonical valuation f : Fm $\longrightarrow \mathbf{L}$

$$f(\varphi) := [\varphi]$$

can be extended to $\overline{f}$: FM$_0 \longrightarrow \overline{\mathbf{L}}$ since $\overline{\mathbf{L}}$ is complete.

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The canonical valuation $f : \mathsf{Fm} \longrightarrow \mathbf{L}$

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can be extended to $\overline{f} : \mathsf{FM}_0 \longrightarrow \overline{\mathbf{L}}$ since $\overline{\mathbf{L}}$ is complete.
If $\mathbf{G}^1\mathbf{LI}_0 \vdash \varphi$ with $\varphi \in \mathsf{Fm}$, then $\overline{f}(\varphi) = \top$ by Soundness.

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The canonical valuation f : Fm $\longrightarrow \mathbf{L}$

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can be extended to $\overline{f}$: FM$_0 \longrightarrow \overline{\mathbf{L}}$ since $\overline{\mathbf{L}}$ is complete.
If $\mathbf{G}^1\mathbf{LI}_0 \vdash \varphi$ with $\varphi \in$ Fm, then $\overline{f}(\varphi) = \top$ by Soundness.
Since $\overline{f} = f$ for Fm (by regularity), we have
$f(\varphi) = [\varphi] = \top$.
That is, $\mathbf{LI} \vdash \varphi$.

Difficulty: the definition of $\overline{f}$ involves

$$\overline{f}(\forall X.\varphi) = \bigwedge_{\xi:\mathsf{Tm}\to\mathbf{L}} \overline{f}_{[X\mapsto\xi]}(\varphi)$$

and Soundness requires comprehension. So does not formalize in inductive theories.

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Key observation

The Ω-rule is valid w.r.t. $\overline{f} : \mathsf{FM}_0 \longrightarrow \overline{\mathbf{L}}$.

The reason is that Ω-rule is "similar" to MacNeille.

$$\frac{\{\ \Delta \Rightarrow \Pi^{\theta}\ \}_{\Delta \Rightarrow^{\mathbf{LI}}_{Y} \varphi^{\theta}(Y)}}{\forall X.\varphi(X) \Rightarrow \Pi} \qquad \frac{\{a \le y\}_{a\ \le\ x}}{x \le y}$$

Motivated by this, we introduce the Ω-valuation $f^\Omega : \mathsf{FM}_0 \longrightarrow \overline{\mathbf{L}}$.

$$
\begin{array}{lcl}
f^\Omega(p(\overline{t})) & = & [p(\overline{t})] \\
f^\Omega(t \in X) & = & [t \in X] \\
f^\Omega(\varphi \to \psi) & = & f^\Omega(\varphi) \to f^\Omega(\psi) \\
f^\Omega(\forall x.\varphi(x)) & = & \bigwedge_{t \in \mathsf{Tm}} f^\Omega(\varphi(t)) \\
f^\Omega(\forall X.\varphi(X)) & = & \bigvee\{[\Delta] \in \mathbf{L} : \Delta \Rightarrow^{\mathbf{LI}}_Y \varphi(Y) \text{ for some } Y\}
\end{array}
$$

Lemma

$\mathbf{G}^1\mathbf{LI}_0$ is sound w.r.t. the Ω-valuation.

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f^\Omega(\forall x.\varphi(x)) & = & \bigwedge_{t \in \mathsf{Tm}} f^\Omega(\varphi(t)) \\
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\end{array}$$

Lemma

$\mathbf{G}^1\mathbf{LI}_0$ is sound w.r.t. the Ω-valuation.

Remark: (Altenkirch-Coquand 01) made a similar observation in the context of λ-calculus, but . . .

Theorem (in $\mathbf{PRA}$, Aehlig 04)

For any 1st order formula φ, if $\mathbf{G}^1\mathbf{LI}_0 \vdash \varphi$, then $\mathbf{PA}$ $(= \mathbf{ID}_0)$ proves "$\mathbf{LI} \vdash \varphi$."

- Should not be confused with a wrong statement that $\mathbf{PA}$ proves "$\mathbf{G}^1\mathbf{LI}_0$ is a conservative extension of $\mathbf{LI}$."
- Each derivation contains finitely many formulas. So you can describe each $f^\Omega(\varphi)$ by a formula, not as a set. No comprehension needed.

We now introduce a uniform framework for MacNeille completion and algebraic cut elimination.

A polarity is $\mathbf{W} = \langle W, W', R \rangle$ where W, W' are sets and $R \subseteq W \times W'$ (Birkhoff 40).
Given $X \subseteq W$ and $Z \subseteq W'$,

$$\begin{array}{rcl} X^{\triangleright} & := & \{z \in W' : \text{ for all } x \in X,\ x \ R \ z\} \\ Z^{\triangleleft} & := & \{x \in W : \text{ for all } z \in Z,\ x \ R \ z\} \end{array}$$

The pair $(\triangleright, \triangleleft)$ forms a Galois connection:

$$X \subseteq Z^{\triangleleft} \quad \Longleftrightarrow \quad X^{\triangleright} \supseteq Z$$

so induces a closure operator on $\wp(W)$:

$$\gamma(X) := X^{\triangleright\triangleleft}.$$

$$\begin{array}{rcl} G(\mathbf{W}) & := & \{X \subseteq W : X = \gamma(X)\} \\ X \cup_\gamma Y & := & \gamma(X \cup Y) \end{array}$$

Lemma

$\mathbf{W}^+ := \langle G(\mathbf{W}), \cap, \cup_\gamma \rangle$ is a complete lattice.
It is a complete Heyting algebra under additional assumptions.

Given a lattice (or Heyting algebra) $\mathbf{A}$,

$$\mathbf{W}_{\mathbf{A}} := \langle A, A, \leq \rangle$$

is a polarity. $X^{\triangleright}$ is the upper bounds of X and $Z^{\triangleleft}$ is the lower bounds of Z. Let $\gamma(a) := \{a\}^{\triangleright\triangleleft}$.

Theorem

$\gamma : \mathbf{A} \longrightarrow \mathbf{W}_{\mathbf{A}}^{+}$ is the MacNeille completion of $\mathbf{A}$.

For example, consider

$$\mathbf{W}_{\mathbb{Q}} := \langle \mathbb{Q}, \mathbb{Q}, \leq \rangle$$

Then for each $X \in G(\mathbf{W})$, $(X, X^{\triangleright})$ is a Dedekind cut. Hence

$$\mathbf{W}_{\mathbb{Q}}^{+} \cong \mathbb{R} \cup \{\pm\infty\}.$$

We now give an algebraic proof to

Theorem

$\mathbf{G}^1\mathbf{LI}_0$ admits cut elimination.

Define a polarity by

$$
\begin{array}{lcl}
\mathbf{W}_{cf} & := & \langle Seq, Cxt, \Rightarrow^{cf} \rangle \\
Seq & := & \mathsf{FM}_0^* \\
Cxt & := & \mathsf{FM}_0^* \times (\mathsf{FM}_0 \cup \{\emptyset\}) \\
\Gamma \Rightarrow^{cf} (\Sigma, \Pi) & \Leftrightarrow & \Gamma, \Sigma \Rightarrow \Pi \text{ is cut-free provable in } \mathbf{G}^1\mathbf{LI}_0.
\end{array}
$$

Fact

$\mathbf{W}_{cf}^+$ is a complete Heyting algebra such that

$$\Gamma \in \varphi^{\lhd} \quad \Longleftrightarrow \quad \Gamma \Rightarrow^{cf} \varphi.$$

One could use the "reducibility candidates" technique as in (Maehara 91) and (Okada 96), but it is too strong for $\mathbf{G}^1\mathbf{LI}_0$. It doesn't formalize in $\mathbf{PA}$.

Ω-valuation $f : \mathsf{FM}_0 \longrightarrow \mathbf{W}^+_{cf}$

$$
\begin{array}{lcl}
f^\Omega(p(\bar{t})) & = & p(\bar{t})^\lhd \\
f^\Omega(t \in X) & = & (t \in X)^\lhd \\
f^\Omega(\varphi \to \psi) & = & f^\Omega(\varphi) \to f^\Omega(\psi) \\
f^\Omega(\forall x.\varphi(x)) & = & \bigcap_{t \in \mathsf{Tm}} f^\Omega(\varphi(t)) \\
f^\Omega(\forall X.\varphi(X)) & = & \forall X.\varphi(X)^\lhd \\
 & = & \{\Delta \in Seq : \Delta \Rightarrow^{cf}_Y \varphi(Y) \text{ for some } Y\}^{\rhd\lhd}
\end{array}
$$

Lemma

$\mathbf{G}^1\mathbf{LI}_0 \vdash \Gamma \Rightarrow \Pi$ implies $f^\Omega(\Gamma) \subseteq f^\Omega(\Pi)$ (Soundness).
$\varphi \in f^\Omega(\varphi) \subseteq \varphi^\lhd$ for any $\varphi \in \mathsf{FM}_0$ (Completeness).

Lemma (recap)

$\mathbf{G}^1\mathbf{LI}_0 \vdash \Gamma \Rightarrow \Pi$ implies $f^\Omega(\Gamma) \subseteq f^\Omega(\Pi)$ (Soundness).
$\varphi \in f^\Omega(\varphi) \subseteq \varphi^\lhd$ for any $\varphi \in \mathsf{FM}_0$ (Completeness).

Now cut elimination for $\mathbf{G}^1\mathbf{LI}_0$ follows easily.

(Proof) Suppose $\mathbf{G}^1\mathbf{LI}_0 \vdash \varphi \Rightarrow \psi$.
Then $f^\Omega(\varphi) \subseteq f^\Omega(\psi)$ by Soundness.
$\varphi \in f^\Omega(\varphi) \subseteq f^\Omega(\psi) \subseteq \psi^\lhd$ by Completeness.
So $\varphi \Rightarrow \psi$ is cut-free provable.

We have shown provability = cut-free provability.
So a fortiori we obtain:

Theorem

$\mathbf{W}^{+}_{cf} \cong \overline{\mathbf{L}_0}$, the MacNeille completion of the Lindenbaum algebra for $\mathbf{G}^1\mathbf{LI}_1$.

algebraic c.elim for $\mathbf{G}^1\mathbf{LI}_0$ = MacNeille compl. + Ω-valuation.

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algebraic c.elim for $\mathbf{G}^1\mathbf{LI}_0$ = MacNeille compl. + Ω-valuation.

By combining it with a syntactic argument based on Ω-rule:

Theorem (in $\mathbf{PRA}$, Aehlig 04)

If $\mathbf{G}^1\mathbf{LI}_n \vdash \varphi$, then $\mathbf{ID}_n$ proves "$\mathbf{G}^1\mathbf{LI}_n \vdash^{cf} \varphi$."

Corollary (in $\mathbf{PRA}$)

1-consistency of $\mathbf{ID}_n$ is equivalent to cut elimination for $\mathbf{G}^1\mathbf{LI}_n$.

For the lambda calculus audience

Why does metatheory matters?

We have been careful in which metatheory the theorem is proved.

Does it matter if one is only interested in the TRUTH?

Yes! Since a proper metatheory consideration often leads to an interesting TRUTH such as

iterated System $\mathbf{T}$ $=$ parameter-free System $\mathbf{F}$.

$\mathbf{T}_0$:= simply typed λ-calculus + basic inductive data types

$$\begin{array}{lll} \texttt{inductive} & \boldsymbol{N} & := \; 0 : \boldsymbol{N} \mid \mathsf{s} : \boldsymbol{N} \Rightarrow \boldsymbol{N} \\ \texttt{inductive} & \boldsymbol{T} & := \; \mathsf{leaf} : \boldsymbol{T} \mid \mathsf{node} : \boldsymbol{T} \Rightarrow \boldsymbol{T} \Rightarrow \boldsymbol{T} \end{array}$$

$\mathbf{T}_1 := \mathbf{T}_0$ + $\mathbf{T}_0$-definable inductive data types

$$\begin{array}{lll} \texttt{inductive} & \boldsymbol{L}(\boldsymbol{N}) & := \; \mathsf{nil} : \boldsymbol{L}(\boldsymbol{N}) \mid \mathsf{cons} : \boldsymbol{N} \Rightarrow \boldsymbol{L}(\boldsymbol{N}) \Rightarrow \boldsymbol{L}(\boldsymbol{N}) \\ \texttt{inductive} & \boldsymbol{O} & := \; 0 : \boldsymbol{O} \mid \mathsf{s} : \boldsymbol{O} \Rightarrow \boldsymbol{O} \mid \mathsf{lim} : (\boldsymbol{N} \Rightarrow \boldsymbol{O}) \Rightarrow \boldsymbol{O} \end{array}$$

$\mathbf{T}_2 := \mathbf{T}_1$ + $\mathbf{T}_1$-definable inductive data types

$\vdots$

$\mathbf{T}_{<\omega} := \bigcup_n \mathbf{T}_n$.

Correspondence between T_n and ID_n

Given a system $\mathbf{X}$ of typed λ-calculus:

$$\mathsf{Rep}(\mathbf{X}) := \text{the functions } f : \mathbb{N} \longrightarrow \mathbb{N} \text{ definable by a term } M^{\boldsymbol{N} \Rightarrow \boldsymbol{N}} \text{ in system } \mathbf{X}.$$

Given a theory $\mathbf{A}$ of arithmetic:

$$\mathsf{Total}(\mathbf{A}) := \text{the functions } f : \mathbb{N} \longrightarrow \mathbb{N} \text{ provably total in theory } \mathbf{A}.$$

Fact

$$\begin{aligned} \mathsf{Rep}(\mathbf{T}_0) &= \mathsf{Total}(\mathbf{PA}) \\ \mathsf{Rep}(\mathbf{T}_n) &= \mathsf{Total}(\mathbf{ID}_n) \\ \mathsf{Rep}(\mathbf{T}_{<\omega}) &= \mathsf{Total}(\mathbf{ID}_{<\omega}) \end{aligned}$$

Type in System $\mathbf{F}$ is defined by:

$$A, B \ ::= \ \alpha \mid A \Rightarrow B \mid \forall \alpha . A.$$

Inductive data types in $\mathbf{T}_{<\omega}$ are all definable in $\mathbf{F}$.

$$\begin{array}{lcl} \boldsymbol{N} & := & \forall \alpha.(\alpha \Rightarrow \alpha) \Rightarrow (\alpha \Rightarrow \alpha) \\ \boldsymbol{O} & := & \forall \alpha.((\boldsymbol{N} \Rightarrow \alpha) \Rightarrow \alpha) \Rightarrow (\alpha \Rightarrow \alpha) \Rightarrow (\alpha \Rightarrow \alpha) \end{array}$$

Theorem

$\mathrm{Rep}(\mathbf{T}_{<\omega}) \subseteq \mathrm{Rep}(\mathbf{F})$.

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Theorem

$\mathrm{Rep}(\mathbf{T}_{<\omega}) \subseteq \mathrm{Rep}(\mathbf{F})$.

Which fragment of System $\mathbf{F}$ exactly corresponds to $\mathbf{T}_{<\omega}$?

Parameter-free System $\mathbf{F}$ (cf. Aehlig 2008)

Type$_n$ ($n \in \mathbb{N} \cup \{-1\}$) is defined by:

$$A_n, B_n \ ::= \ \alpha \mid A_n \Rightarrow B_n \mid \forall\alpha.A_{n-1}, \qquad (\mathsf{Fv}(A_{n-1}) \subseteq \{\alpha\})$$

$$\begin{array}{lcll}
\boldsymbol{N} & := & \forall\alpha.(\alpha \Rightarrow \alpha) \Rightarrow (\alpha \Rightarrow \alpha) & \in \mathsf{Type}_0 \\
\boldsymbol{T} & := & \forall\alpha.(\alpha \Rightarrow \alpha \Rightarrow \alpha) \Rightarrow (\alpha \Rightarrow \alpha) & \in \mathsf{Type}_0 \\
\boldsymbol{L}(\boldsymbol{N}) & := & \forall\alpha.(\boldsymbol{N} \Rightarrow \alpha \Rightarrow \alpha) \Rightarrow (\alpha \Rightarrow \alpha) & \in \mathsf{Type}_1 \\
\boldsymbol{O} & := & \forall\alpha.((\boldsymbol{N} \Rightarrow \alpha) \Rightarrow \alpha) \Rightarrow (\alpha \Rightarrow \alpha) \Rightarrow (\alpha \Rightarrow \alpha) & \in \mathsf{Type}_1 \\
\boldsymbol{L}(\beta) & := & \forall\alpha.(\beta \Rightarrow \alpha \Rightarrow \alpha) \Rightarrow (\alpha \Rightarrow \alpha) & \notin \mathsf{Type} \\
 & & \forall\beta.(\boldsymbol{L}(\beta) \Rightarrow \beta) \Rightarrow \beta & \notin \mathsf{Type}
\end{array}$$

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\end{array}
$$

$\mathbf{F}^p_n$:= System $\mathbf{F}$ with types restricted to Type$_n$.
$\mathbf{F}^p_{<\omega} := \bigcup_n \mathbf{F}^p_n$.

$\mathbf{F}^p_{-1}$ is just simply typed lambda calculus.
$\mathbf{F}^p_0$ is studied by (Altenkirch-Coquand 2001).

Parameter-free F and iterated T

Theorem (Akiyoshi-T. 16)

$\mathbf{ID}_{n+1} \vdash \mathsf{SN}(\mathbf{F}^p_n)$.
$\mathbf{ID}_n \vdash \Phi\text{-}\mathsf{SN}(\mathbf{F}^p_n)$ for any finite $\Phi \subseteq \mathsf{Type}_n$.

The proof consists of

- inductive definition of SN-terms + Ω-rule
- Tait's computability predicate + "Ω-valuation"

The 2nd statement implies: for every closed term $M : \boldsymbol{N} \Rightarrow \boldsymbol{N}$ of $\mathbf{F}^p_n$,

$$\mathbf{ID}_n \vdash \forall x \exists y.\ \text{“}M\underline{x} =_\beta \underline{y}\text{”}, \quad \text{hence} \quad \mathsf{Rep}(\mathbf{F}^p_n) \subseteq \mathsf{Total}(\mathbf{ID}_n).$$

Theorem (Altenkirch-Coquand 01, Aehlig 08)

$\mathsf{Rep}(\mathbf{F}^p_0) = \mathsf{Rep}(\mathbf{T}_0) = \mathsf{Total}(\mathbf{PA})$.
$\mathsf{Rep}(\mathbf{F}^p_n) = \mathsf{Rep}(\mathbf{T}_n) = \mathsf{Total}(\mathbf{ID}_n)$.
$\mathsf{Rep}(\mathbf{F}^p_{<\omega}) = \mathsf{Rep}(\mathbf{T}_{<\omega}) = \mathsf{Total}(\mathbf{ID}_{<\omega})$.

For the nonclassical logics audience

Recall:

Theorem (Harding-Bezhanishvili 04)

$\mathcal{HA}$ and $\mathcal{BA}$ are the only nontrivial subvarieties of $\mathcal{HA}$ closed under MacNeille completions.

On the other hand, one finds abundant of examples in substructural logics and associated residuated lattices.

Theorem (Ciabattoni-Galatos-T. 12)

- There are infinitely many varieties of residuated lattices closed under MacNeille completions.
- So there are infinitely many substructural logics that admit algebraic cut elimination.

For intermediate logics, a useful framework is hypersequent calculus. Associated completion is hyper-MacNeille completion.

Theorem (Ciabattoni-Galatos-T. 08, 17)

- There are infinitely many subvarieties of $\mathcal{HA}$ closed under hyper-MacNeille completions.
- So there are infinitely many intermediate logics that admit algebraic cut elimination in hypersequent calculi.

On the other hand, there are also counterexamples for cut elimination/completion in substructural logics. That is WHY substructural logics are interesting!

Theorem

- There is an MV algebra (Chang's chain) which cannot be embedded into a complete MV algebra.
- That is, $\mathcal{MV}$ is not closed under any completion (cf. Litak-Kowalski 06 for more).

- Hence Łukasiewicz infinite-valued logic cannot be conservatively extended with infinitary $\bigwedge$.
- That is, Ł has NO "good" proof system (although some exist . . .).

- Ω-rule is valid for the MacNeille completion of the Lindenbaum algebra.
- This leads to algebraic cut elimination for $\mathbf{G}^1\mathbf{LI}_1$ based on MacNeille completion + Ω-valuation.

Target system	Fragments	Full higher-order logics
Algebraic proof	MacNeille + Ω-valuation	MacNeille + reducibility candidates
Syntactic proof	Ω-rule	Takeuti's Conjecture

$$
\begin{array}{lcl}
\text{1-consistency of } \mathbf{ID}_n & = & \text{cut-elimination for } \mathbf{G}^1\mathbf{LI}_n \\
\text{iterated System } \mathbf{T} & = & \text{parameter-free System } \mathbf{F}.
\end{array}
$$