Equivalence of Inductive Definitions and Cyclic Proofs under Arithmetic

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MLA 2018
Kanazawa, Japan
March 5–9, 2018

Introduction

Inductive definition
- Least fixedpoint
- Production rules

LKID
- Classical Martin-Löf's system of inductive definitions
- Elimination rule of inductive predicates

$\mathbf{CLKID}^{\omega}$
- Cyclic proof system (Brotherston 2006, Brotherston et al 2011)

Brotherston-Simpson conjecture
- Equivalence of **LKID** and $\mathbf{CLKID}^{\omega}$
- Was open: $\mathbf{CLKID}^{\omega}$ to **LKID**
- False in general (Berardi and Tatsuta 2017)

Result: Equivalence between **LKID** and $\mathbf{CLKID}^{\omega}$ under **PA**

Ideas:
- Path relation for stage numbers
- Podelski-Rybalchenko Theorem for induction principle

Inductive Definitions

Inductive predicate symbols P

Production rules

$$Eg. \qquad \frac{}{N0} \qquad \frac{Nx}{Nsx}$$

Least fixed point

Martin-Löf's Inductive Definition System LKID

Introduction rules

$$\frac{}{\Gamma \vdash N0, \Delta} \qquad \frac{\Gamma \vdash Nx, \Delta}{\Gamma \vdash Nsx, \Delta}$$

Elimination rule

$$\frac{\Gamma \vdash F0, \Delta \quad \Gamma, Fx \vdash Fsx, \Delta \quad \Gamma, Ft \vdash \Delta}{\Gamma, Nt \vdash \Delta}$$

- describes mathematical induction principle
$(\forall x.Nx \wedge Fx \to Fsx) \to (\forall x.Nx \to Fx)$

Cyclic Proof System CLKID^ω

System obtained from $\mathbf{LKID}$ by
(1) replacing elimination rules by case rules
(2) allowing a bud as an open assumption and requiring a companion (some corresponding sequent of the same form inside the proof figure) for each bud
(3) requiring global trace condition (when unfolding to an infinite path, it infinitely progresses)

Case rule for N

$$\frac{\Gamma, t = 0 \vdash \Delta \quad \Gamma, t = sx, Nx \vdash \Delta}{\Gamma, Nt \vdash \Delta}$$

Cyclic Proof System CLKID^{ω} (cont.)

Production rules

$$\frac{}{E0} \qquad \frac{Ox}{Esx} \qquad \frac{Ex}{Osx}$$

Proof

$$\dfrac{\dfrac{\dfrac{}{x = 0 \vdash \mathbf{N}x} \qquad \dfrac{\dfrac{\dfrac{\color{red}{(b)\ Ox \vdash \mathbf{N}x}}{x = sx', Ox' \vdash \mathbf{N}x'}\,(Subst)(Wk)}{x = sx', Ox' \vdash \mathbf{N}sx'}\,(\mathbf{N}\ R)}{x = sx', Ox' \vdash \mathbf{N}x}\,(=R)}{\color{blue}{(a)\ Ex \vdash \mathbf{N}x}}\,(\text{Case } E) \qquad \dfrac{\dfrac{\dfrac{\dfrac{\color{blue}{(a)\ Ex \vdash \mathbf{N}x}}{x = sx', Ex' \vdash \mathbf{N}x'}\,(Subst)(Wk)}{x = sx', Ex' \vdash \mathbf{N}sx'}\,(\mathbf{N}\ R)}{x = sx', Ex' \vdash \mathbf{N}x}\,(=R)}{\color{red}{(b)\ Ox \vdash \mathbf{N}x}}\,(\text{Case } O)}{Ex \vee Ox \vdash \mathbf{N}x}\,(\vee L)$$

(a) and (b) denote the bud-companion relation

Brotherston-Simpson Conjecture

Conjecture (Brotherston 2006, Brotherston et al 2011). Provability in $\mathbf{CLKID}^{\omega}$ is the same as that in $\mathbf{LKID}$

- Known: $\mathbf{LKID}$ to $\mathbf{CLKID}^{\omega}$ (Brotherston 2006, Brotherston et al 2011)
- Was open: $\mathbf{CLKID}^{\omega}$ to $\mathbf{LKID}$
- Proved to be false in general (Berardi and Tatsuta 2017)
- Proved to be true when the inductive predicates in a system are only the natural number predicate N (Simpson 2017)

This talk: Provability in $\mathbf{CLKID}^{\omega}$ is the same as that in $\mathbf{LKID}$ if both systems contain $\mathbf{PA}$
- includes Simpson's 2017 result

Addition of Peano arithmetic

$\mathbf{CLKID}^{\omega} + \mathbf{PA}$ and $\mathbf{LKID} + \mathbf{PA}$

- adding Peano arithmetic
- Function symbols $0, s, +, \times$, ordinary predicate symbol $<$
- Inductive predicate symbol $\mathbf{N}$, productions for $\mathbf{N}$
- Axioms

$$
\begin{aligned}
&\vdash \mathbf{N}x \to sx \neq 0, \\
&\vdash \mathbf{N}x \wedge \mathbf{N}y \to sx = sy \to x = y, \\
&\vdash \mathbf{N}x \to x + 0 = x, \\
&\vdash \mathbf{N}x \wedge \mathbf{N}y \to x + sy = s(x + y), \\
&\vdash \mathbf{N}x \to x \times 0 = 0, \\
&\vdash \mathbf{N}x \wedge \mathbf{N}y \to x \times sy = x \times y + x, \\
&\vdash x < y \leftrightarrow \mathbf{N}x \wedge \mathbf{N}y \wedge \exists z(\mathbf{N}z \wedge x + sz = y).
\end{aligned}
$$

Notations

- Sequence of numbers $\langle t_0, \ldots, t_n \rangle$
- u-th element of sequence $(t)_u$ (starting from 0-th)
- Pt or $t \in P$ for $P(t)$

Stage Numbers

Inductive atomic formula
- its predicate symbol is an inductive predicate symbol

Stage transformation for inductive atomic formula:
We transform $P(t)$ into $\exists v P'(t,v)$
- P' is a new inductive predicate symbol
- $P'(t,v)$ means that the element t comes into P at stage v
- v is called a stage number
- $P(t)$ and $\exists v P'(t,v)$ are equivalent

Stage transformation for production rules

$$\frac{Q_1\mathbf{u}_1 \quad \dots \quad Q_n\mathbf{u}_n \quad P_1\mathbf{t}_1 \quad \dots \quad P_m\mathbf{t}_m}{P\mathbf{t}}$$

is transformed into

$$\frac{Q_1\mathbf{u}_1 \quad \dots \quad Q_n\mathbf{u}_n \quad v > v_1 \quad P'_1\mathbf{t}_1 v_1 \quad \dots \quad v > v_m \quad P'_m\mathbf{t}_m v_m \quad \mathbf{N}v}{P'\mathbf{t}v}$$

$$Eg. \quad \frac{\mathbf{N}v}{\mathbf{N}'0v} \qquad \frac{\mathbf{N}v \quad v > v_1 \quad \mathbf{N}'xv_1}{\mathbf{N}'sxv} \qquad \frac{\mathbf{N}v \quad v > v_1 \quad E'xv_1}{O'sxv}$$

Main Theorem

Theorem 1 (Proof Transformation from $\mathbf{CLKID}^\omega + \mathbf{PA}$ to $\mathbf{LKID} + \mathbf{PA}$)
Let $\Sigma = \{0, s, +, \times, <, Q_1, \ldots, Q_m, N, P_1, \ldots, P_n\}$,
$\Phi = \{N, P_1, \ldots, P_n\}$,
$\Sigma' = \Sigma \cup \{N', P'_1, \ldots, P'_n\}$,
$\Phi' = \Phi \cup \{N', P'_1, \ldots, P'_n\}$.
If $\mathbf{CLKID}^\omega + \mathbf{PA} + (\Sigma, \Phi)$ *proves* $\Gamma \vdash \Delta$,
then $\mathbf{LKID} + \mathbf{PA} + (\Sigma', \Phi')$ *proves* $\Gamma \vdash \Delta$.

For a given cyclic proof in $\mathbf{CLKID}^\omega + \mathbf{PA}$ we will construct a proof of the same conclusion in $\mathbf{LKID} + \mathbf{PA}$

By using some conservativity lemma for $\mathbf{LKID}$, this theorem shows:

Corollary 2 (Equivalence of $\mathbf{LKID} + \mathbf{PA}$ and $\mathbf{CLKID}^\omega + \mathbf{PA}$)
Let $\Sigma = \{0, s, +, \times, <, Q_1, \ldots, Q_m, N, P_1, \ldots, P_n, P'_1, \ldots, P'_n\}$,
$\Phi = \{N, P_1, \ldots, P_n, P'_1, \ldots, P'_n\}$.
If $\mathbf{CLKID}^\omega + \mathbf{PA} + (\Sigma, \Phi)$ *proves* $\Gamma \vdash \Delta$,
then $\mathbf{LKID} + \mathbf{PA} + (\Sigma, \Phi)$ *proves* $\Gamma \vdash \Delta$.

These show the conjecture is true under Peano arithmetic

Main Idea

For a given cyclic proof in $\mathbf{CLKID}^\omega + \mathbf{PA}$
- For each companion take a subproof with the root being the companion.
- Forget bud-companion relations (buds become open assumption)

$$\dfrac{\overline{x = 0 \vdash \mathbf{N}x} \qquad \dfrac{\dfrac{\dfrac{Ox \vdash \mathbf{N}x}{x = sx', Ox' \vdash \mathbf{N}x'}}{x = sx', Ox' \vdash \mathbf{N}sx'}}{x = sx', Ox' \vdash \mathbf{N}x}}{Ex \vdash \mathbf{N}x} \text{(Case } E\text{)} \qquad \dfrac{\dfrac{\dfrac{Ex \vdash \mathbf{N}x}{x = sx', Ex' \vdash \mathbf{N}x'}}{x = sx', Ex' \vdash \mathbf{N}sx'}}{\dfrac{x = sx', Ex' \vdash \mathbf{N}x}{Ox \vdash \mathbf{N}x}} \text{(Case } O\text{)}$$

We will define some appropriate relation $>_\Pi$ on a sequence of numbers
- $\text{Ind}(>_\Pi)$ is provable in $\mathbf{LKID} + \mathbf{PA}$
- in the stage-number transformation of each subproof, the sequence of the stage numbers of any assumption is less than that of the conclusion by $>_\Pi$

Main Idea 1: Path Relation

For a path π from a companion J_2 to an assumption J_1, define $x \tilde{>}_\pi y$ by:
- x (or y) is the sequence of length being the number of primed inductive atomic formulas (such as $P'(t,v)$) in the antecedent of J_2 (or J_1),
- $(x)_p > (y)_q$ (or $(x)_p = (y)_q$) if there is a progressing (or non-progressing) trace from the p-th primed inductive atomic formula of J_2 to the q-the primed inductive atomic formula of J_1,

We define path relation $\langle x_0, x \rangle >_\pi \langle y_0, y \rangle$ by: - $x \tilde{>}_\pi y$
- x_0, y_0 are the companion numbers of the bottom and top sequents

B_0 - the set of a path from the conclusion to an assumption in these subproofs
B - the set of all finite compositions of paths in B_0

$\{>_\pi \ \mid \pi \in B\}$ is finite
- $>_\pi$ is described by finite information ($>$ or $=$ among the elements and the companion numbers)

Define $>_\Pi$ as the union of $\{>_\pi \ \mid \pi \in B\}$.

Main Idea 2: Induction Principle

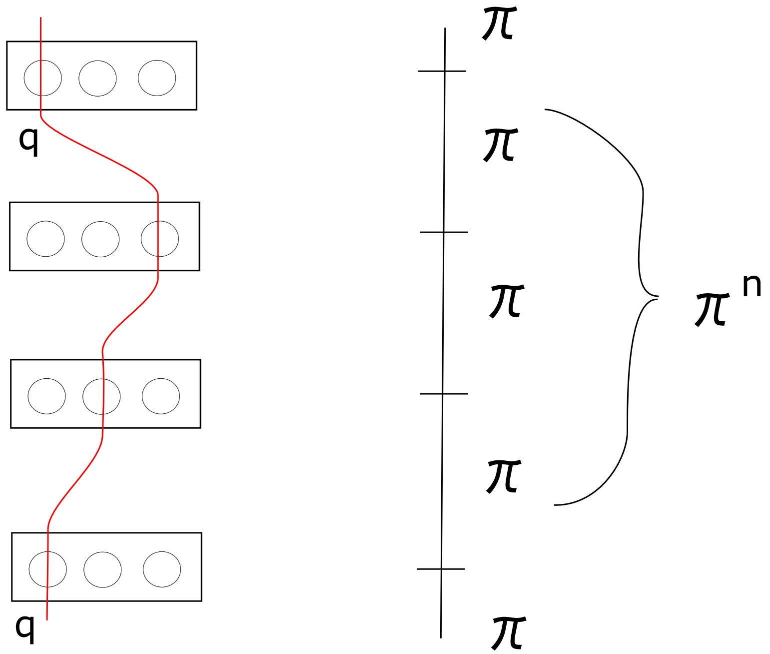
Induction principle with ($>_\Pi$): Ind($>_\Pi$) $\equiv (\forall x.(\forall y <_\Pi x.Fy) \to Fx) \to \forall x.Fx$

We will show Ind($>_\Pi$) is provable in $\mathbf{LKID} + \mathbf{PA}$ by:
(1) For each $\pi \in B$, there is n such that Ind($<^n_\pi$)
(2) arithmetical infinite Ramsey theorem
(3) Podelski-Rybalchenko termination theorem for induction schema

The global trace condition gives (1). (3) is proved by (2). Combining (1) and (3), we will obtain Ind($>_\Pi$).

Main Idea 2: Induction Principle (cont.)

Proof sketch. (1) Consider the infinite path $\pi\pi\pi\ldots$ in the infinite unfolding of Π. By the global trace condition, in general, since the numbers of primed inductive atomic formulas in Π are limited, there are n, m, q such that some progressing trace passes the q-the primed inductive atomic formula in the top sequent of π^m and the q-the primed inductive atomic formula in the top sequent of π^{m+n}. If $x >_\pi^n y$, then $x >_{\pi^n} y$, which implies $(x)_q > (y)_q$. By mathematical induction with this, Ind($>_\pi^n$) is provable in $\mathbf{LKID} + \mathbf{PA}$.



Main Idea 2: Induction Principle (cont.)

(3) Podelski-Rybalchenko termination theorem for induction schema: if transition invariant $>_\Pi$ is a finite union of relations $>_\pi$ such that $\mathrm{Ind}(>_\pi^n)$ is provable for some n, then $\mathrm{Ind}(>_\Pi)$ is provable.

We can show it by replacing well-foundedness by induction principle in the original proof in [Podelski et al 2004]. Since their proof used infinite Ramsey theorem, we need infinite Ramsey theorem in $\mathbf{LKID} + \mathbf{PA}$, which is obtained by (2).

(2) Arithmetical infinite Ramsey theorem: given coloring formulas we can effectively construct a formula such that Peano arithmetic shows the formula describes an infinite sequence of the same color.

We can prove it by formalizing an ordinary proof of infinite Ramsey theorem in Peano arithmetic.

Conclusion

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