Minimalist Algebraic Local Set Theory

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Abstract of our talk

- What is Algebraic Set Theory?
- Why developing it?
- Algebraic Set Theory for the extensional level of the **Minimalist Foundation**
- future work

What is Algebraic Set Theory?

origin:

Algebraic Set Theory, A. Joyal and I. Moerdijk, CUP, 1995

= *Categorical set theory*

see: http://www.phil.cmu.edu/projects/ast/whyast.html

What is Algebraic Set Theory?

in [JM95]

categorical models for ZFC and IZF axiomatic set theories

key point:

Von Neumann Universe **V** = Initial ZF-algebra

Peculiarity of Algebraic Set Theory

notion of model of a set theory

via algebraic universal properties

=

derive set existence (including universes)

via categorical properties

Our goal

Algebraic Set Theory

for

the **Minimalist Foundation** (**MF**)

actually for its extensional level in [M.'09]

intended as the minimalist set theory

where to formalize constructive mathematics

according to [M.-Sambin'05]

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an example of categorical model for **MF**

in the *next talk by Samuele Maschio*

---

it employs a realizability interpretation

in joint work by us with

Hajime Ishihara and *Thomas Streicher*

to appear in 2018 in *Archive for Mathematical Logic* *essence of our talk*

internal language of topoi

= local set theory

algebraic set theory

minimalist local set theory

= **predicative** and **minimalist**

version of topos

*Why using Category Theory (**CT**) in model theory ?*

- **CT** provide models type theories and their proof-terms
  in an easy/NON-trivial/intuitive way

- **CT** provides a framework to relate a calculus and its models
  in a way stronger than usual soundness-completeness relation
  via the **Internal Language correspondence**
  even for USUAL proof-irrelevant logical systems

Example where Categorical Modelling is necessary

No classical set theoretic notion of model

for Coquand's Calculus of Constructions

(the one implemented in **Coq**)

extending

standard set-theoretic model of typed lambda calculus

because

"Polymorphism is not set-theoretic" di John Reynolds, in "Semantics of Data Types", 1984

Volume 173 of the series Lecture Notes in Computer Science pp 145-156

and

All small complete categories are preordered (proof-irrelevant).

Killer application of **Category Theory** *in logic*

Lawvere's hyperdoctrines

=

notion of funtorial model

for classical/intuitionistic logic

even with equality

enjoy an **Internal Language correspondence**

with respect to the corresponding logic

while

Tarskian/ Complete Boolean valued models

for classical logic

and

Topological/Complete Heyting valued models

for intuitionistic logic

DO NOT enjoy it

Lawvere's hyperdoctrines for first order classical predicate calculus

= Boolean hyperdoctrines

i.e. suitable functors towards cat of Boolean algebras

$$D \quad : \mathcal{C}^{OP} \longrightarrow \mathbf{Boole}$$

$$A \mapsto D(A)$$

Boolean algebra

$$f\colon A \to B \mapsto D(B) \to D(B)$$

Boolean algebra homomorphism

+ existential and universal adjunctions

Lawvere's hyperdoctrines for first order intuitionistic predicate calculus

= Heyting hyperdoctrines

i.e. suitable functors towards cat of Heyting algebras

$$
\begin{array}{llll}
D & : \mathcal{C}^{OP} & \longrightarrow & \mathbf{Hey} \\
 & A & \mapsto & D(A) \\
 & & & \text{Heyting algebra} \\
 & f : A \to B & \mapsto & D(B) \to D(B) \\
 & & & \text{Heyting algebra homomorphism}
\end{array}
$$

+ existential and universal adjunctions

Novelty of Lawvere's hyperdoctrines

Connectives and quantifiers

are modelled as left/right adjoints

i.e. via *universal properties*

[Bill Lawvere, "Adjointness in Foundations", (TAC), Dialectica 23 (1969), 281-296]

Logical connectives as Adjoints

also called Galois connections

falsum constant	left adjoint to singleton constant functor
true constant	right adjoint to singleton constant functor
conjunction	right adjoint to diagonal functor
conjunction	left adjoint to diagonal functor
implication of ϕ, i.e. $\phi \rightarrow (-)$	right adjoint to the conjunction functor with ϕ
intuitionistic negation $\neg(-)$	left adjoint to itself towards the opposite category
classical negation $\neg(-)$	is ALSO right adjoint to itself towards the opposite category (to get the excluded middle principle)

Quantifiers as Adjoints

- Universal Quantifiers are Right Adjoints to Weakening:

$$\frac{\psi \leq_{[x]} \forall y.(\phi[z/y])}{Weak_z(\psi) \leq_{[x,z]} \phi}$$

$Weak_y(\psi) = \psi$

= ψ does NOT depend from y.

- Existential Quantifiers are Left Adjoints to Weakening

$$\frac{\exists y.(\phi[z/y]) \leq_{[x]} \psi}{\phi \leq_{[x,z]} Weak_z(\psi)}$$

- + Beck-Chevalley conditions

Advantage of categorical modelling

easy proof of **soundness+completeness theorem**
NO need of NON-constructive principles

syntactic hyperdoctrine for **classical**/**intuitionistic** logic

=

initial boolean/Heyting hyperdoctrine

in the category of corresponding hyperdoctrines

+ related homomorphisms

$$LT_{\mathbf{LC}}\colon \quad Cont^{OP} \longrightarrow \mathbf{Boole}$$

$$\Gamma \mapsto LT(\Gamma)$$

context of variable assumptions — Lindenbaum algebra of formulas with variables in Γ

$$[\,t_1/x_1, \ldots, t_n/x_n\,]\colon \Delta \to \Gamma \quad \mapsto \quad LT_{\mathbf{LC}}([t_1/x_1, \ldots, t_n/x_n])$$

n-tuple of term substitutions — substitution morphism

Killer application of Categorical Logic

Lawvere's hyperdoctrines are related to the corresponding logic

NOT ONLY via usual **soundness+ completeness** relation:

$\Gamma \vdash \phi$ derivable in the logic

iff

$(\Gamma \vdash \phi)^{I}$ holds in each model interpretation $(-)^{I}$

but also.... via the **internal language correspondence**!!

To establish an internal language correspondence

Given a calculus τ

- organize its theories (= τ + axioms) into a category with translations

$$\mathrm{Th}(\tau)$$

- organize a class of its models into a category

$$\mathrm{Mod}(\tau)$$

- Define a functor extracting the **internal theory** out of a model

$$\begin{array}{lccc} Int\colon & \mathrm{Mod}(\tau) & \rightarrow & \mathrm{Th}(\tau) \\ & M & \mapsto & Int(\mathsf{M}) \end{array}$$

internal theory of M

- Define a model out of a theory of τ:

$$Syn\colon \quad \mathrm{Th}(\tau) \to \mathrm{Mod}(\tau)$$
$$T \mapsto Syn(\mathsf{M})$$

syntactic model of T

such that

for any model M

$$\mathsf{M} \simeq Syn(Int(\mathsf{M}))$$

for any theory T

$$\mathsf{T} \simeq Int(Syn(\mathsf{T}))$$

logic	complete models	**internal language correspondence**
classical propositional logic	Boolean algebras	yes
intuitionistic propositional logic	Heyting algebras	yes
classical predicate logic	Complete Boolean valued models	NO
	Boolean hyperdoctrines	yes
intuitionistic predicate logic	Complete Heyting algebras	NO
	Heyting hyperdoctrines	yes

Other examples of **internal languages**

- **first example**: Benabou-Mitchell internal language of Lawvere-Tierney elementary topoi

  as a many-sorted INTUITIONISTIC logic

- **internal language** as a dependent type theory à la Martin-Löf

  is given for many categorical structures:

  lex categories

  regular categorie

  locally cartesian closed categories

  pretopoi

  elementary topoi...

  in [M.05] "Modular correspondence between dependent type theories and categories.."

Internal language *of a topos is a local set theory*

J. Bell *Toposes and Local Set Theories: An Introduction*. Clarendon Press, Oxford, 1988

Local Set Theory	=	Set theory with typed variables
	=	Set theory + Type Theory

axiomatic set theory for ex: Friedman's IZF	local set theory of topoi
first order predicate logic ⇓ untyped variables	many sorted logic with *sorts*=types=sets ⇓ typed variables
powerset axiom	powerset type
subsets as sets	subsets as elements of powerset
extensional equality of sets	extensional equality of subsets
one kind of functions: functional relations	two kinds of functions: functional relations + functional typed terms (=base morphisms) whilst in bijection by unique choice rule

in the local set theory of topoi + natural numbers object

type of natural numbers	Nat
type of subsets of natural numbers	$\mathcal{P}(Nat)$
membership	from a subset
	$U \in \mathcal{P}(Nat)$
	we can form
	$x \varepsilon U \; prop \; [x \in Nat]$
comprehension axiom	from a proposition
	$\phi(x) \; prop \; [x \in Nat]$
	we can form
	$\{ x \in Nat \mid \phi(x) \} \in \mathcal{P}(Nat)$
	s.t. it true that
	$\mathbf{n} \varepsilon \{ x \in Nat \mid \phi(x) \} \Leftrightarrow \phi(\mathbf{n})$

example of functional typed terms: given a term

$$f(x, y) \in B \; [z \in C, x \in A]$$

we can form

$$\lambda x.f(x) \in A \;\rightarrow B \; [z \in C]$$

DEPENDENT typed internal language *for topoi*

Internal dependent type theory à la Martin-Löf of elementary topoi in [M'05, M.PhD thesis'98]

in

M.E.M "*Modular correspondence between dependent type theories and categorical universes including pretopoi and topoi." MSCS, 2005*

following [Bell'88]

internal dependent typed language of topoi in [M'05]	=	Local Set Theory
	=	Set theory with **dependent** typed variables
	=	Set theory + **Dependent** Type Theory

Internal languages *of topoi*

Benabou/Mitchell language	Internal language in [M'05]
many sorted logic with simple types	Dependent type theory with dependent types
sets (=types) $\neq$ propositions	propositions as mono sets(=types)
propositions as terms of the classifier $\mathcal{P}(1)$	$\mathcal{P}(1)$ classifies mono sets up to equiprovability

In both languages

sets $\neq$subsets

subsets of a set A =elements of the powerset of A

comprehension axiom holds

Notion of proposition in a topos in [M'05]

In the **internal dependent type theory** of topoi

a *proposition* P is a *monoset*: if we derive $P\ set$ and a proof p $p \in Eq(P, w, z)\ [w \in P, z \in P]$
a *predicate* $P(x)$ is a *mono dependent set*: if we derive $P(x) set\ [x \in A]$ and a proof p $p \in Eq(P, w, z)\ [x \in A, w \in P(x), z \in P(x)]$

similar to HoTT

but in an extensional type theory

What categorical models for the **Minimalist Foundation**?

a brief recap of why developing

the **Minimalist Foundation**

to formalize constructive mathematics

Plurality of constructive foundations ⇒ need of a minimalist foundation

	classical	constructive
	ONE standard	NO standard
impredicative	Zermelo-Fraenkel set theory	internal theory of topoi Calculus of Inductive Constructions
predicative	Feferman's explicit maths	Aczel's Constructive Zermelo-Fraenkel set th. Martin-Löf's type theory Feferman's constructive expl. maths

what common core ??

Need of a MINIMALIST FOUNDATION

Plurality of constructive foundations (often mutual incompatible)

$$\Downarrow$$

Need of a core foundation where to find common proofs

and doing constructive REVERSE mathematics!!

our (M.-Sambin's proposal): adopt the MINIMALIST FOUNDATION

from [M.-Sambin'05], [M.09]

Plurality of constructive foundations $\Rightarrow$ *need of a minimalist foundation*

	classical	constructive
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impredicative	Zermelo-Fraenkel set theory	internal theory of topoi Calculus of Inductive Constructions
predicative	Feferman's explicit maths	Aczel's Constructive Zermelo-Fraenkel set th. Martin-Löf's type theory Feferman's constructive expl. maths

the MINIMALIST FOUNDATION is a common core

What **foundation** *for constructive mathematics?*

(j.w.w. G. Sambin)

a FORMAL Constructive Foundation should include

extensional LANGUAGE of abstract maths

as usual set theoretic language

↓ interpreted in

intensional trustable base

for an INTERACTIVE prover

↓ interpreted in

a PROGRAMMING LANGUAGE

acting as a realizability model

(for proofs-as-programs extraction)

our notion of **constructive foundation**

a three-level foundation

= a two-level foundation + a realizability level

<table>
<tr><td rowspan="3">PURE

Foundation</td><td>extensional level (used by mathematicians to do their proofs)</td></tr>
<tr><td>⇓ interpreted via a QUOTIENT model</td></tr>
<tr><td>intensional level (language of computer-aided formalized proofs)</td></tr>
</table>

⇓

realizability level (used by computer scientists to extract programs) *our notion of* **constructive foundation** *employs different languages*

language of *(LOCAL) AXIOMATIC SET THEORY*	for extensional level
language of *CATEGORY THEORY*	algebraic structure to link intensional/extensional levels via a quotient completion
language of TYPE THEORY	for intensional level
computational language	for realizability level

the pure TWO-LEVEL structure of the Minimalist Foundation

from [Maietti'09]

- its intensional level

  = a PREDICATIVE VERSION of the Calculus of Inductive Constructions

  = a FRAGMENT of Martin-Löf's intensional type theory

- its extensional level

  is a PREDICATIVE LOCAL set theory

  (NO choice principles)

we use CATEGORY THEORY

to express the link between extensional/intensional levels:

use
notion of ELEMENTARY QUOTIENT COMPLETION
(in the language of CATEGORY THEORY)
relative to a suitable **Lawvere's doctrine**

in:

[M.E.M.-Rosolini'13] "Quotient completion for the foundation of constructive mathematics", Logica Universalis

[M.E.M.-Rosolini'13] "Elementary quotient completion", Theory and Applications of Categories

see *Fabio Pasquali's talk*

*What realizability level for **MF**?*

Martin-Löf's type theory

or

an extension of Kleene realizability

of **intensional level of MF**+ Axiom of Choice + Formal Church's thesis

as in

H. Ishihara, M.E.M., S. Maschio, T. Streicher

Consistency of the Minimalist Foundation with Church's thesis and Axiom of Choice

in *AML* 2018.

Differences with Martin-Löf's type theory

Both levels of **MF** are dependent type theories
based on intensional/extensional versions
of *Martin-Löf's type theory*
(for short **MLTT**)

but with remarkable differences:

intensional level of **MF**	**MLTT**
distinction sets/collections	all types are sets
primitive propositions	propositions-as-sets
distinction between small **propositions** and **propositions**	
elimination of **propositions** only towards **propositions**	general elimination
NO rule/axiom of unique choice	YES rule/axiom of unique choice
NO rule/axiom of choice	YES rule/axiom of choice
universe of small **propositions**	universe of small sets

Differences between intensional/extensional levels of **MF**

Both levels of **MF** in [M'09]
are **dependent type theories**

How do they differ??

intensional level of **MF**	extensional level of **MF**
universe of small propositions	power-collection of subsets of 1
universe of small propositional functions on a set	powercollection of a set
proof-relevant propositions	**proof-irrelevant** propositions
Martin-Löf's constructors on sets with only β-conversions	Martin-Löf's constructors on sets with β and η-conversions
proof-relevant Identity type eliminating only towards propositions	**proof-irrelevant** Identity type à la Martin-Löf
decidable definitional equality	undecidable definitional equality
	effective quotient sets

Differences topoi/extensional level of **MF**

They both are

both are local set theory
including
extensional Martin-Löf's 1st-order constructors of sets

dependent type theory of topoi in [M'05]	**extensional** level of **MF** in [M'09]
all types are sets	distinction sets/collections
propositions as mono sets	**primitive** propositions/predicates
	small propositions/propositions
YES axiom/rule of unique choice	NO axiom/rule of unique choice

two notions of function in **MF**

a *primitive notion* of type-theoretic function

$$f(x) \in B \; [x \in A]$$

$\neq$ (syntactically)

notion of functional relation

$$\forall x \in A \; \exists! y \in B \; R(x, y)$$

$\Rightarrow$ NO axiom of unique choice in **MF**

Axiom of unique choice

$$\forall x \in A\, \exists! y \in B\, R(x,y) \quad \longrightarrow \quad \exists f \in A \to B\, \forall x \in A\, R(x, f(x))$$

turns a functional relation into a type-theoretic function.

$\Rightarrow$ identifies the two distinct notions...

Essence of the extensional level of **MF**

the extensional level of **MF**

had been designed

as a **minimalist** and **predicative** version

of the **internal** dependent type theory

of topoi in [M'05]

which we know is a local set theory from [Bell'88]

by adopting the distinction small maps within a **category**

from Algebraic Set Theory in [Joyal-Moerdijk'95]

What is the algebraic set theory for the intensional level of **MF** *?*

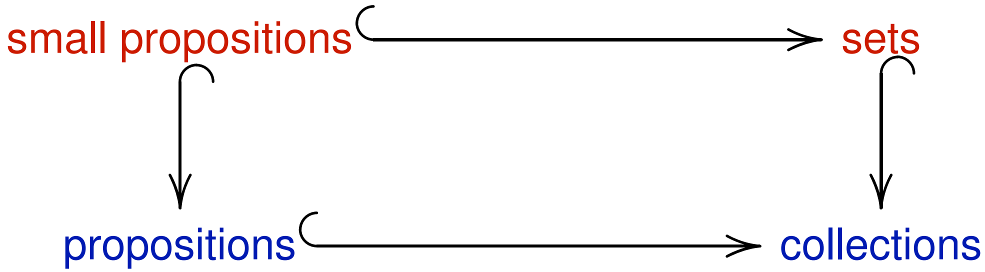
Cartmell's contextual categories	=	algebraic axiomatization adapted to the intensional level of **MF** in [M.'09]

Notion of categorical model for the extensional level of **MF**

a minimalist and predicative generalization of the notion of elementary topos

Minimalist Algebraic Local Set theory = minimalist predicative elementary topos = **MF**-topos (for short)	Algebraic Set Theory
ambient category of collections	ambient category of collections
small maps defined via Benabou's fibrations with **primitive** fibrations for **propositions** and small **propositions**	small maps defined via axioms
universe via a classifier object which is a collection	universe via a classifier object which is small for **IZF**, **ZF**

ENTITIES in the Minimalist Foundation



are represented by

a **MF**-topos defined as

a *finite limit* category of collections $\mathcal{C}$

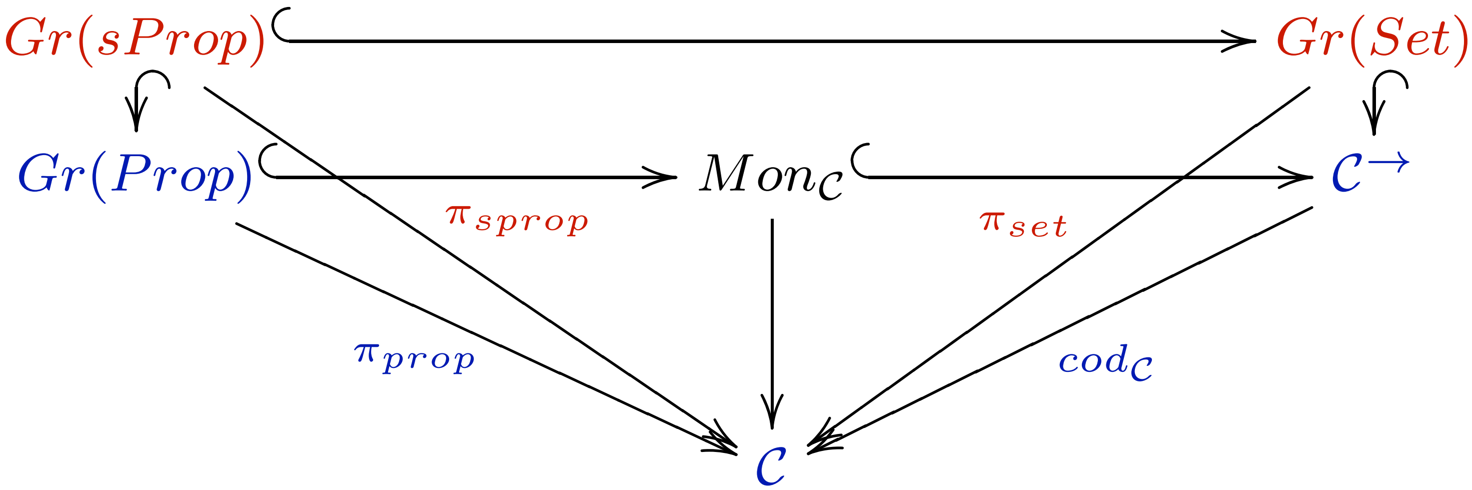
together with three **fibrations à la Benabou** over $\mathcal{C}$

representing the other types

Minimalist Elementary topos

A **MF**-**Elementary topos** is a tuple of full sub-fibered categories of the codomain fibration of a lex category $\mathcal{C}$ (meant to be collections)

$$(\mathcal{C}, \pi_{set}, \pi_{prop}, \pi_{sprop})$$



where all the inclusion are cartesian FULL embeddings

modelling MF-types.

examples of MF-elementary toposes

- The syntactic one from the extensional level of **MF**

  (pure minimalist one!)

- A predicative version of Hyland's Effective Topos (next talk)

  (with unique choice).

- the setoid model over

  Martin-Löf's type theory with one universe

  (with unique choice)

More examples of **MF***-toposes...??*

We need to make a

minimalist and predicative tripos-to-topos construction

via the FREE ALGEBRAIC construction

called **Elementary Quotient Completion** of an **Elementary doctrine**

introduced in

[M.-Rosolini'13] "Quotient completion for the foundation of constructive mathematics", Logica Universalis

[M.-Rosolini'13] ""Elementary quotient completion", Theory and Applications of Categories.

see *Fabio Pasquali's talk*

the **Elementary quotient completion**

gives an algebraic axiomatization of the quotient/setoid model

used to interpret the extensional level of **MF**

into its intensional one

in [M'09]

in terms of its universal properties.

Future work

- Relate our notion of minimalist predicative version of topos
  i.e. the notion of **MF**-Elementary topos
  to **Moerdijk-Palmgren-van den Berg**'s notion of predicative topos
  and to algebraic set theory for CZF.

- Build a **boolean** **MF**- topos
  with no unique choice
  in one of Feferman's predicative theories.

- Investigate peculiar aspects of Homotopy Type Theory in **MF**:
  look for weak factorization systems within the intensional level of **MF**.