A remark on predicate extensions of intuitionistic logic

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Hallmarks of Constructive Reasoning: Disjunction and Existence Properties

Definition

A logic $\mathbf{L}$ is said to have the disjunction property (DP), if for every $A \vee B$:

$$\mathbf{L} \vdash A \vee B \Rightarrow \mathbf{L} \vdash A \text{ or } \mathbf{L} \vdash B.$$

A logic $\mathbf{L}$ is said to have the existence property (EP), if for every $\exists x A(x)$:

$$\mathbf{L} \vdash \exists x A(x) \Rightarrow \text{there exists a } v \text{ such that } \mathbf{L} \vdash A(v).$$

- DP and EP are regarded as distinguishing features and characteristics of constructivity of **Int**.

$$\text{“}\exists x A(x) = \bigvee_{u \in U} A(u)\text{”}$$

Predicate extensions of intuitionistic logic

We consider Predicate Extensions of Intuitionistic logic (PEI's).
Int: Intuitionistic predicate logic,

Definition

(1) The propositional part of **L** is the set $\{A : A$ is a propositional formula and provable in $\mathbf{L}\}$.
(2) **L** is said to be a PEI, if its propositional part coincides with that of **Int**.

By the definition, PEI's must resemble **Int** in some logical characters; and PEI's must NOT resemble **Int** in other logical characters.

Int is the logic of constructivity.
We deal with two properties related to constructivity of **Int** ;

- Disjunction property (DP),
- Existence property (EP).

Relations in PEI's?

		DP	
		Yes	No
EP	Yes	**Int** etc.	**Int** + CD $+GJ \vee Z$ etc. S. 2013(?)
	No	**Int** + F etc. Nakamura ('83), Minari ('83-'86)	**Int** + F + Lin^* etc.

$$
\begin{aligned}
CD: &\quad \forall x(p(x) \vee q) \to \forall x p(x) \vee q \ , \\
GJ: &\quad \bigwedge_{i=0}^{2} \left((q_i \to \bigvee_{j \neq i} q_j) \to \bigvee_{j \neq i} q_j\right) \to \bigvee_{i=0}^{2} q_i \ , \\
Z: &\quad \exists x p(x) \to \forall x p(x) \ . \\
F: &\quad \exists x(p(x) \to \forall y p(y)) \ , \\
Lin^*: &\quad \forall x \forall y((p(x) \to p(y)) \vee (p(y) \to p(x)))
\end{aligned}
$$

How Many PEI's?

		DP	
		Yes	No
EP	Yes	2^ω	2^ω
	No	2^ω	2^ω

Theorem

(1) There exist 2^ω PEI's having DP but lacking EP.
(2) There exist 2^ω PEI's lacking DP and EP.

How?

By constructing a concrete sequence of axiom schemata

$$X_i \quad (i < \omega)$$

such that;

1. for $I \subseteq \omega$, $\mathbf{Int} + \{X_i \,;\, i \in I\}$ is a PEI lacking EP,
2. for each $i < \omega$, $\mathbf{Int} + \{X_j \,;\, j \neq i\} \nvdash X_i$,
   - $\Longrightarrow$ For $I, J \subseteq \omega$,
     $I = J$ if and only if $\mathbf{Int} + \{X_k \,;\, k \in I\} = \mathbf{Int} + \{X_k \,;\, k \in J\}$.
3. for $I \subseteq \omega$, $\mathbf{Int} + \{X_i \,;\, i \in I\}$ enjoys DP.

No.1.

$\mathbf{A}$: a finite & strongly compact Heyting algebra; the 2nd greatest $= \star_{\mathbf{A}}$.
Take a finite set $\{p_a \,;\, a \in \mathbf{A}\}$ of propositional variables.

$$\delta(\mathbf{A}) = \{p_{a \cup_{\mathbf{A}} b} \equiv (p_a \vee p_b), p_{a \cap_{\mathbf{A}} b} \equiv (p_a \wedge p_b) \,;\, a, b \in \mathbf{A}\}$$
$$\bigcup \{p_{a \to_{\mathbf{A}} b} \equiv (p_a \to p_b), p_{a \to_{\mathbf{A}} 0_{\mathbf{A}}} \equiv (\neg p_a) \,;\, a, b \in \mathbf{A}\},$$

$$J(\mathbf{A}) : \ \left(\bigwedge \delta(\mathbf{A}) \right) \to p_{\star_{\mathbf{A}}},$$

$$QJ(\mathbf{A}) : \ \exists v \{ \left(\bigwedge \delta(\mathbf{A}) \right) [p_{\star_{\mathbf{A}}} := F] \to (P(v) \to \forall y P(y)) \}.$$

$$(F : \ \exists x (P(x) \to \forall y P(y)). \ )$$

Lemma

If S is a set of finite and strongly compact Heyting algebras having at least three elements, then $\mathbf{Int} + \{QJ(\mathbf{A}) : \mathbf{A} \in S\}$ is a PEI lacking EP.

No.2.

Fact (*cf.* Wroński '73)

B: a Heyting algebra,
$E(\mathbf{B})$: the propositional logic determined by **B**.

$E(\mathbf{B}) \nvdash J(\mathbf{A})$ if and only if **A** is embeddable into a quotient algebra of **B**.

By making use of the algebraic Kripke sheaf semantics (S. '99), we have:

Lemma

A, **B**: finite and strongly compact Heyting algebras in which there exist exactly three elements having no incomparable elements.

$L(\mathcal{K}(\mathbf{B})) \nvdash QJ(\mathbf{A})$ if and only if **A** is embeddable into a quotient alg. of **B**.

$\mathcal{K}(\mathbf{B})$: a special algebraic Kripke sheaf determined by **B**,
$L(\mathcal{K}(\mathbf{B}))$: the super-intuitionistic predicate logic determined by $\mathcal{K}(\mathbf{B})$.

No.2. (Cont'd)

Suppose we have a sequence $\{\mathbf{A}_i\}_{i<\omega}$ of Heyting algebras such that

- each $\mathbf{A}_i$ is a finite and strongly compact Ha in which there exist exactly three elements having no incomparable elements,
- for $i, j < \omega$ $(i \neq j)$, $\mathbf{A}_i$ is not embeddable into any quotient alg. of $\mathbf{A}_j$.

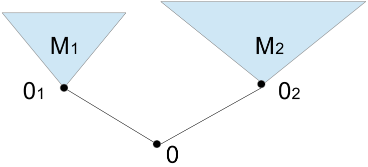
Lemma

For each $i < \omega$, $\mathbf{Int} + \{QJ(\mathbf{A}_j)\,; j \neq i\} \not\ni QJ(\mathbf{A}_i)$.

No.3.

Definition

$(\mathbf{M}_i, D_i, \models_i)$ $(i = 1, 2)$; Kripke models with the least elements 0_1 and 0_2.
The pointed join model of them:



$$D^{\uparrow}(a) = \begin{cases} D_1(0_1) \times D_2(0_2) & \text{if } a = 0, \\ D_1(a) \times D_2(0_2) & \text{if } a \in \mathbf{M}_1, \\ D_1(0_1) \times D_2(a) & \text{if } a \in \mathbf{M}_2. \end{cases}$$

Lemma

If A_i is not true in $(\mathbf{M}_i, D_i, \models_1)$ $(i = 1, 2)$,
then $A_1 \vee A_2$ is not true in the pointed join model of them.

No.3. (Cont'd)

Definition

A formula A is axiomatically true in a Kripke model,
if every instance of A is true in it.

Lemma

If the axiomtic truth of A is preserved under pointed-join-model construction, then $\mathbf{Int} + A$ has DP.

Lemma

$\mathbf{A}$: a finite and strongly compact Heyting algebra $\mathbf{A}$ with $|\mathbf{A}| \geq 3$.
The axiomtic truth of $QJ(\mathbf{A})$ is preserved under pointed-join-model construction. Thus, $\mathbf{Int} + QJ(\mathbf{A})$ has DP.

Completing the proof

By modifying Wroński('73)'s technique,
we constructed a concrete sequence $\{\mathbf{A}_i\}_{i<\omega}$ of Heyting algebras such that

- each $\mathbf{A}_i$ is a finite and strongly compact Ha in which there exist exactly three elements having no incomparable elements,
- for $i, j < \omega$ $(i \neq j)$, $\mathbf{A}_i$ is not embeddable into any quotient alg. of $\mathbf{A}_j$,

Therefore, we have a concrete sequence of axiom schemata

$$QJ(\mathbf{A}_i) \quad (i < \omega)$$

such that;

1. for $I \subseteq \omega$, $\mathbf{Int} + \{QJ(\mathbf{A}_i); i \in I\}$ is a PEI lacking EP,
2. for each $i < \omega$, $\mathbf{Int} + \{QJ(\mathbf{A}_j)\,; j \neq i\} \nvdash QJ(\mathbf{A}_i)$,
   - $\Longrightarrow$ For $I, J \subseteq \omega$,
     $I = J$ if and only if $\mathbf{Int} + \{QJ(\mathbf{A}_k)\,; k \in I\} = \mathbf{Int} + \{QJ(\mathbf{A}_k)\,; k \in J\}$.
3. for $I \subseteq \omega$, $\mathbf{Int} + \{QJ(\mathbf{A}_i)\,; i \in I\}$ enjoys DP.

Concluding Remarks

$$??\ \exists x A(x) = \bigvee_{u \in U} A(u) ??$$

1. We constructed a recursively enumerable set of concrete axiom schemata. By adding these schemata to **Int**, we obtained a continuum of PEI's each of which has DP but lacks EP.
2. We have four continua of PEI's:
    - "with EP and DP,"
    - "without EP and DP,"
    - "with DP but without EP," and
    - "with EP but without DP."

   Other than the last one, Three of them can be obtained by constructing a recursively enumerable set of concrete axioms.

   But for the last continuum: "with EP but without DP," we do not have such a set of axiom schemata, as yet.

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