Hyper Natural Deduction for Gödel Logic a natural deduction system for parallel reasoning[1]

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Motivation

- Arnon Avron: *Hypersequents, Logical Consequence and Intermediate Logics for Concurrency* Ann.Math.Art.Int. 4 (1991) 225-248

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  - *The name "communication rule" hints, of course, at a certain intuitive interpretation that we have of it as corresponding to the idea of exchanging information between two multiprocesses: [...]*
- General aim: provide Curry-Howard style correspondences for parallel computation, starting from logical systems with good intuitive algebraic / relational semantics.

Setting the stage

Setting the stage

IL

Setting the stage

IL λ

Setting the stage

IL $\Leftrightarrow$ ND λ

Setting the stage

IL $\Leftrightarrow$ ND λ

Gentzen '34

Setting the stage

IL $\Leftrightarrow$ ND λ

introduction rule

$$\frac{A \qquad B}{A \wedge B}$$

elimination rule

$$\frac{A \wedge B}{A}$$

Setting the stage

IL $\Leftrightarrow$ ND λ

introduction rule	elimination rule
$\dfrac{A \qquad B}{A \wedge B}$	$\dfrac{A \wedge B}{A}$
$\dfrac{\dfrac{[A]}{B}}{A \to B}$	$\dfrac{A \qquad A \to B}{B}$

Setting the stage

IL $\Leftrightarrow$ ND λ

introduction rule	elimination rule
$\dfrac{A \qquad B}{A \wedge B}$	$\dfrac{A \wedge B}{A}$
$\dfrac{\dfrac{[A]}{B}}{A \rightarrow B}$	$\dfrac{A \qquad A \rightarrow B}{B}$

normalisation

Setting the stage

IL $\Leftrightarrow$ LJ $\Leftrightarrow$ ND $\qquad$ λ

Gentzen '34

Setting the stage

IL $\Leftrightarrow$ LJ $\Leftrightarrow$ ND λ

Sequent	Axiom	Rules
$\Gamma \Rightarrow A$	$A \Rightarrow A$	$\dfrac{\Gamma \Rightarrow A \qquad \Delta \Rightarrow B}{\Gamma, \Delta \Rightarrow A \wedge B}$

Setting the stage

IL $\Leftrightarrow$ LJ $\Leftrightarrow$ ND $\qquad$ λ

Sequent	Axiom	Rules
$\Gamma \Rightarrow A$	$A \Rightarrow A$	$\dfrac{\Gamma \Rightarrow A \qquad \Delta \Rightarrow B}{\Gamma, \Delta \Rightarrow A \wedge B}$

$$(\text{cut})\ \frac{\Gamma \Rightarrow A \qquad \Pi, A \Rightarrow B}{\Gamma, \Pi \Rightarrow B}$$

Setting the stage

IL $\Leftrightarrow$ LJ $\Leftrightarrow$ ND λ

Sequent	Axiom	Rules
$\Gamma \Rightarrow A$	$A \Rightarrow A$	$\dfrac{\Gamma \Rightarrow A \qquad \Delta \Rightarrow B}{\Gamma, \Delta \Rightarrow A \wedge B}$

$$(\text{cut})\ \frac{\Gamma \Rightarrow A \qquad \Pi, A \Rightarrow B}{\Gamma, \Pi \Rightarrow B}$$

cut elimination – consistency

Setting the stage

$$\text{IL} \quad \Leftrightarrow \quad \text{LJ} \quad \Leftrightarrow \quad \text{ND} \quad \overset{\text{Curry Howard}}{\Longleftrightarrow} \quad \lambda$$

Every proof system hides a model of computation.

Setting the stage

IL $\Leftrightarrow$ LJ $\Leftrightarrow$ ND $\overset{\text{Curry Howard}}{\Longleftrightarrow}$ λ

GL

Setting the stage

$$\mathrm{IL} \Leftrightarrow \mathrm{LJ} \Leftrightarrow \mathrm{ND} \xLeftrightarrow{\text{Curry Howard}} \lambda$$

Gödel '32, Dummett '59

GL

Setting the stage

IL $\Leftrightarrow$ LJ $\Leftrightarrow$ ND $\overset{\text{Curry Howard}}{\Longleftrightarrow}$ λ

Gödel '32, Dummett '59

GL IL + LIN $(A \to B) \vee (B \to A)$

Setting the stage

IL $\Leftrightarrow$ LJ $\Leftrightarrow$ ND $\overset{\text{Curry Howard}}{\Longleftrightarrow}$ λ

GL

Gödel '32, Dummett '59

IL + LIN $(A \to B) \vee (B \to A)$

Logic of Linear Kripke Frames

Setting the stage

$\mathrm{IL} \Leftrightarrow \mathrm{LJ} \Leftrightarrow \mathrm{ND} \overset{\text{Curry Howard}}{\Longleftrightarrow} \lambda$

$\mathrm{GL} \Leftrightarrow \mathrm{HLK}$

Avron '91

Setting the stage

$$\mathrm{IL} \Leftrightarrow \mathrm{LJ} \Leftrightarrow \mathrm{ND} \overset{\text{Curry Howard}}{\Longleftrightarrow} \lambda$$

$$\mathrm{GL} \Leftrightarrow \mathrm{HLK}$$

Hypersequent

$$\Gamma_1 \Rightarrow \Delta_1 \mid \ldots \mid \Gamma_n \Rightarrow \Delta_n$$

Setting the stage

$$\mathrm{IL} \Leftrightarrow \mathrm{LJ} \Leftrightarrow \mathrm{ND} \overset{\text{Curry Howard}}{\Longleftrightarrow} \lambda$$

$$\mathrm{GL} \Leftrightarrow \mathrm{HLK}$$

$$\frac{\Gamma \Rightarrow A \qquad \Delta \Rightarrow B}{\Gamma \Rightarrow B \mid \Delta \Rightarrow A}$$

Setting the stage

$$\mathrm{IL} \quad \Leftrightarrow \quad \mathrm{LJ} \quad \Leftrightarrow \quad \mathrm{ND} \quad \overset{\text{Curry Howard}}{\Longleftrightarrow} \quad \lambda$$

$$\mathrm{GL} \quad \Leftrightarrow \quad \mathrm{HLK} \qquad\qquad (\text{com})\ \frac{\Gamma \Rightarrow A \qquad \Delta \Rightarrow B}{\Gamma \Rightarrow B \mid \Delta \Rightarrow A}$$

Setting the stage

$$\text{IL} \Leftrightarrow \text{LJ} \Leftrightarrow \text{ND} \overset{\text{Curry Howard}}{\Longleftrightarrow} \lambda$$

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Avron '91: *Communication between agents*

Setting the stage

$$\text{IL} \Leftrightarrow \text{LJ} \Leftrightarrow \text{ND} \overset{\text{Curry Howard}}{\Longleftrightarrow} \lambda$$

$$\text{GL} \Leftrightarrow \text{HLK} \qquad (\text{com})\ \frac{\Gamma \Rightarrow A \qquad \Delta \Rightarrow B}{\Gamma \Rightarrow B \mid \Delta \Rightarrow A}$$

Fermüller '08: Lorenzen style dialogue games, ...

Setting the stage

$$\text{IL} \quad \Leftrightarrow \quad \text{LJ} \quad \Leftrightarrow \quad \text{ND} \quad \overset{\text{Curry Howard}}{\Longleftrightarrow} \quad \lambda$$

$$\text{GL} \quad \Leftrightarrow \quad \text{HLK} \qquad\qquad (\text{com})\ \frac{\Gamma \Rightarrow A \qquad \Delta \Rightarrow B}{\Gamma \Rightarrow B \mid \Delta \Rightarrow A}$$

hyper sequent calculi for various logics

Setting the stage

$$\text{IL} \Leftrightarrow \text{LJ} \Leftrightarrow \text{ND} \overset{\text{Curry Howard}}{\Longleftrightarrow} \lambda$$

$$\text{GL} \Leftrightarrow \text{HLK} \qquad \text{(com)}\ \frac{\Gamma \Rightarrow A \qquad \Delta \Rightarrow B}{\Gamma \Rightarrow B \mid \Delta \Rightarrow A}$$

deep inference

Setting the stage

$$\text{IL} \Leftrightarrow \text{LJ} \Leftrightarrow \text{ND} \overset{\text{Curry Howard}}{\Longleftrightarrow} \lambda$$

$$\text{GL} \Leftrightarrow \text{HLK} \qquad\qquad ?$$

Setting the stage

$$\mathrm{IL} \Leftrightarrow \mathrm{LJ} \Leftrightarrow \mathrm{ND} \overset{\text{Curry Howard}}{\Longleftrightarrow} \lambda$$

$$\mathrm{GL} \Leftrightarrow \mathrm{HLK} \Leftrightarrow \mathrm{HND} \qquad ?$$

Setting the stage

$$\mathrm{IL} \Leftrightarrow \mathrm{LJ} \Leftrightarrow \mathrm{ND} \overset{\text{Curry Howard}}{\Longleftrightarrow} \lambda$$

$$\mathrm{GL} \Leftrightarrow \mathrm{HLK} \Leftrightarrow \mathrm{HND} \Longleftrightarrow ? \quad ?$$

Setting the stage

$$\mathrm{IL} \Leftrightarrow \mathrm{LJ} \Leftrightarrow \mathrm{ND} \overset{\text{Curry Howard}}{\Longleftrightarrow} \lambda$$

$$\mathrm{GL} \Leftrightarrow \mathrm{HLK} \Leftrightarrow \mathrm{HND} \Longleftrightarrow ? \quad ?$$

today's topic

Previous work

Hirai, FLOPS 2012

A Lambda Calculus for Gödel-Dummett Logics Capturing Waitfreedom

- change of both syntax and semantics
- different calculus

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Baaz, Ciabattoni, Fermüller 2000

A Natural Deduction System for Intuitionistic Fuzzy Logic
(will be discussed later)

Wishlist

Properties we want to have:

(semi) local

- construction of deductions:
  apply ND inspired rules to extend a HND deductions
- modularity of deductions:
  reorder/restructure deductions
- analyticity (sub-formula property, ...)

Wishlist

Properties we want to have:

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- construction of deductions:
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normalisation

- procedural normalisation via conversion steps

Natural Deduction rules

$$\wedge\text{-}i\ \frac{A \qquad B}{A \wedge B} \qquad\qquad \wedge\text{-}e\ \frac{A \wedge B}{A} \qquad \frac{A \wedge B}{B}$$

$$\vee\text{-}i\ \frac{A}{A \vee B} \qquad \frac{B}{A \vee B} \qquad\qquad \vee\text{-}e\ \frac{A \vee B \qquad \begin{array}{c}[A]\\ C\end{array} \qquad \begin{array}{c}[B]\\ C\end{array}}{C}$$

$$\to\text{-}i\ \frac{\begin{array}{c}[A]\\ B\end{array}}{A \to B} \qquad\qquad \to\text{-}e\ \frac{A \qquad A \to B}{B}$$

$$\bot_I\ \frac{\bot}{A}$$

Hypersequent Calculus

Hypersequent: $\Gamma_1 \Rightarrow A_1 \mid \ldots \mid \Gamma_n \Rightarrow A_n$

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Hypersequent: $\Gamma_1 \Rightarrow A_1 \mid \ldots \mid \Gamma_n \Rightarrow A_n$
Some rules:

$$\to,l\ \frac{\Gamma \Rightarrow A \mid \mathcal{H} \qquad \Gamma, B \Rightarrow C \mid \mathcal{H}'}{\Gamma, A \to B \Rightarrow C \mid \mathcal{H} \mid \mathcal{H}'}$$

$$\to,r\ \frac{\Gamma, A \Rightarrow B \mid \mathcal{H}}{\Gamma \Rightarrow A \to B \mid \mathcal{H}}$$

Hypersequent Calculus

Hypersequent: $\Gamma_1 \Rightarrow A_1 \mid \ldots \mid \Gamma_n \Rightarrow A_n$
Some rules:

$$\rightarrow,l\ \frac{\Gamma \Rightarrow A \mid \mathcal{H} \qquad \Gamma, B \Rightarrow C \mid \mathcal{H}'}{\Gamma, A \rightarrow B \Rightarrow C \mid \mathcal{H} \mid \mathcal{H}'}$$

$$\rightarrow,r\ \frac{\Gamma, A \Rightarrow B \mid \mathcal{H}}{\Gamma \Rightarrow A \rightarrow B \mid \mathcal{H}}$$

$$com\ \frac{\Gamma_1 \Rightarrow A_1 \mid \mathcal{H} \qquad \Gamma_2 \Rightarrow A_2 \mid \mathcal{H}'}{\Gamma_1 \Rightarrow A_2 \mid \Gamma_2 \Rightarrow A_1 \mid \mathcal{H} \mid \mathcal{H}'}$$

$$split\ \frac{\Pi, \Gamma \Rightarrow A \mid \mathcal{H}}{\Pi \Rightarrow A \mid \Gamma \Rightarrow A \mid \mathcal{H}}$$

Linearity in LJ

$$\frac{}{\Rightarrow (A \to B) \vee (B \to A)}$$

Linearity in LJ

$$\dfrac{\dfrac{}{\Rightarrow A \to B}}{\Rightarrow (A \to B) \vee (B \to A)}$$

Linearity in LJ

$$\dfrac{\dfrac{\dfrac{???}{A \Rightarrow B}}{\Rightarrow A \to B}}{\Rightarrow (A \to B) \vee (B \to A)}$$

Linearity in HLK

$$\frac{}{\Rightarrow (A \to B) \vee (B \to A)}$$

Linearity in HLK

$$\dfrac{\overline{\Rightarrow (A \to B) \lor (B \to A) \mid \Rightarrow (A \to B) \lor (B \to A)}}{\Rightarrow (A \to B) \lor (B \to A)}\ \mathrm{EC}$$

Linearity in HLK

$$
\dfrac{\dfrac{\overline{\Rightarrow (A \to B) \lor (B \to A) \mid \Rightarrow B \to A}}{\Rightarrow (A \to B) \lor (B \to A) \mid \Rightarrow (A \to B) \lor (B \to A)}\ \lor\text{-r}}{\Rightarrow (A \to B) \lor (B \to A)}\ \text{EC}
$$

Linearity in HLK

$$\dfrac{\dfrac{\dfrac{}{\Rightarrow A \rightarrow B \mid \ \Rightarrow B \rightarrow A}}{\Rightarrow (A \rightarrow B) \vee (B \rightarrow A) \mid \ \Rightarrow B \rightarrow A}\ \vee\text{-r}}{\dfrac{\Rightarrow (A \rightarrow B) \vee (B \rightarrow A) \mid \ \Rightarrow (A \rightarrow B) \vee (B \rightarrow A)}{\Rightarrow (A \rightarrow B) \vee (B \rightarrow A)}\ \text{EC}}\ \vee\text{-r}$$

Linearity in HLK

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\Rightarrow A \to B \mid B \Rightarrow A}}{\Rightarrow A \to B \mid\ \Rightarrow B \to A}\ \to\text{-r}}{\Rightarrow (A \to B) \lor (B \to A) \mid\ \Rightarrow B \to A}\ \lor\text{-r}}{\dfrac{\Rightarrow (A \to B) \lor (B \to A) \mid\ \Rightarrow (A \to B) \lor (B \to A)}{\Rightarrow (A \to B) \lor (B \to A)}\ \text{EC}}\ \lor\text{-r}$$

Linearity in HLK

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A \Rightarrow B \mid B \Rightarrow A}}{\Rightarrow A \to B \mid B \Rightarrow A}\ \to\text{-r}}{\Rightarrow A \to B \mid\ \Rightarrow B \to A}\ \to\text{-r}}{\Rightarrow (A \to B) \lor (B \to A) \mid\ \Rightarrow B \to A}\ \lor\text{-r}}{\dfrac{\Rightarrow (A \to B) \lor (B \to A) \mid\ \Rightarrow (A \to B) \lor (B \to A)}{\Rightarrow (A \to B) \lor (B \to A)}\ \text{EC}}\ \lor\text{-r}
$$

Linearity in HLK

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \Rightarrow A \quad B \Rightarrow B}{A \Rightarrow B \mid B \Rightarrow A}\,com}{\Rightarrow A \to B \mid B \Rightarrow A}\,\to\text{-r}}{\color{red}{\Rightarrow A \to B \mid\ \Rightarrow B \to A}}\,\to\text{-r}}{\Rightarrow (A \to B) \vee (B \to A) \mid\ \Rightarrow B \to A}\,\vee\text{-r}}{\dfrac{\Rightarrow (A \to B) \vee (B \to A) \mid\ \Rightarrow (A \to B) \vee (B \to A)}{\color{red}{\Rightarrow (A \to B) \vee (B \to A)}}\,\text{EC}}\,\vee\text{-r}
$$

BCF System

Models hyper sequents in natural deduction by combining deductions in ND with a new operator $|$.

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Example linearity: From

$$\text{From} \quad \frac{\{A\}}{A} \quad \text{and} \quad \frac{\{B\}}{B}$$

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Models hyper sequents in natural deduction by combining deductions in ND with a new operator $|$.

Example linearity: From

$$\text{From} \quad \frac{\{A\}}{A} \quad \text{and} \quad \frac{\{B\}}{B}$$

$$\text{one derives} \quad \dfrac{\{A\}}{\textit{(com)}\,\dfrac{A}{B}} \quad \Bigg| \quad \dfrac{\{B\}}{\textit{(com)}\,\dfrac{B}{A}}$$

BCF System

Models hyper sequents in natural deduction by combining deductions in ND with a new operator $|$.
Example linearity: From

$$\text{From} \quad \frac{\{A\}}{A} \quad \text{and} \quad \frac{\{B\}}{B}$$

$$\text{one derives} \qquad (com)\,\frac{\dfrac{\{A\}}{A}}{B} \;\Bigg|\; (com)\,\frac{\dfrac{\{B\}}{B}}{A}$$

$$\text{then} \qquad \frac{(com)\,\dfrac{\dfrac{\{A\}}{A}}{B}}{A \to B} \;\Bigg|\; \frac{(com)\,\dfrac{\dfrac{\{B\}}{B}}{A}}{B \to A} \qquad \text{etc}$$

Discussion of the BCF system

- direct translation from HLK
- inductive definition
- easy to translate proofs back and forth
- normalisation only via translation to HLK

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Not a solution to our problem!

Our approach to Hyper Natural Deduction

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$$(\text{com})\ \frac{\Gamma \Rightarrow A \qquad \Delta \Rightarrow B}{\Gamma \Rightarrow B \mid \Delta \Rightarrow A}$$

$$\begin{array}{c}\Gamma \\ \vdots \\ A\end{array}$$

Our approach to Hyper Natural Deduction

$$(\text{com})\ \frac{\Gamma \Rightarrow A \qquad \Delta \Rightarrow B}{\Gamma \Rightarrow B \mid \Delta \Rightarrow A}$$

$$\begin{array}{ccc} \Gamma & \qquad & \Delta \\ \vdots & & \vdots \\ A & & B \end{array}$$

Our approach to Hyper Natural Deduction

$$(\text{com})\ \frac{\Gamma \Rightarrow A \qquad \Delta \Rightarrow B}{\Gamma \Rightarrow B \mid \Delta \Rightarrow A}$$

$$\begin{array}{ccc} \Gamma & & \Delta \\ \vdots & & \vdots \\ \mathit{com}\ \dfrac{A}{B} & & B \end{array}$$

OUR APPROACH TO HYPER NATURAL DEDUCTION

$$(\text{com})\ \frac{\Gamma \Rightarrow A \qquad \Delta \Rightarrow B}{\Gamma \Rightarrow B \mid \Delta \Rightarrow A}$$

$$\begin{array}{cc} \Gamma & \Delta \\ \vdots & \vdots \end{array}$$

$$com\ \frac{A}{B} \quad \overline{com}\ \frac{B}{A}$$

Our approach to Hyper Natural Deduction

$$(\text{com})\ \frac{\Gamma \Rightarrow A \qquad\qquad \Delta \Rightarrow B}{\Gamma \Rightarrow B \mid \Delta \Rightarrow A}$$

$$\begin{array}{c}\Gamma\\ \vdots\\ com\ \dfrac{A}{B}\end{array} \quad \begin{array}{c}\Delta\\ \vdots\\ \overline{com}\ \dfrac{B}{A}\end{array}$$

Our approach to Hyper Natural Deduction

$$(\text{com})\ \frac{\Gamma \Rightarrow A \qquad \Delta \Rightarrow B}{\Gamma \Rightarrow B \mid \Delta \Rightarrow A} \qquad\qquad \begin{array}{c}\Gamma\\ \vdots\\ A\end{array} \qquad \begin{array}{c}\Delta\\ \vdots\\ B\end{array}$$

$$com\ \frac{A}{B} \quad \overline{com}\ \frac{B}{A}$$

- consider sets of derivation trees
- divide communication (and split) into two dual parts
- search for minimal set of conditions that provides sound and complete deduction system

Rules of HNGL

Rules for NJ plus

$$\dfrac{\begin{array}{c}{}^{k}[\Gamma],\Delta\\ \vdots\\ A\end{array}}{A}\; r\!:\!{}^{k}\,\mathrm{Spt}_{\Gamma,\Delta}$$

Rules of HNGL

Rules for NJ plus

$$
r\!:^{k}\mathrm{Spt}_{\Gamma,\Delta}\ \dfrac{\begin{matrix}{}^{k}[\Gamma],\Delta\\ \vdots\\ A\end{matrix}}{A}
\qquad\qquad
r\!:\mathrm{Com}_{A,B}\ \dfrac{\begin{matrix}\Gamma\\ \vdots\\ A\end{matrix}}{B}
$$

Rules of HNGL

Rules for NJ plus

$$r{:}^{k}\,\mathrm{Spt}_{\Gamma,\Delta}\ \dfrac{\begin{matrix} {}^{k}[\Gamma],\Delta \\ \vdots \\ A \end{matrix}}{A} \qquad\qquad r{:}\,\mathrm{Com}_{A,B}\ \dfrac{\begin{matrix} \Gamma \\ \vdots \\ A \end{matrix}}{B}$$

$$r{:}\,\mathrm{Ctr}\ \dfrac{\begin{matrix} \Gamma \\ \vdots \\ A \end{matrix} \qquad \begin{matrix} \Delta \\ \vdots \\ A \end{matrix}}{A}$$

Rules of HNGL

Rules for NJ plus

$$r{:}^{k}\,\mathrm{Spt}_{\Gamma,\Delta}\ \dfrac{\begin{array}{c}{}^{k}[\Gamma],\Delta\\ \vdots\\ A\end{array}}{A} \qquad\qquad r:\mathrm{Com}_{A,B}\ \dfrac{\begin{array}{c}\Gamma\\ \vdots\\ A\end{array}}{B}$$

$$r:\mathrm{Ctr}\ \dfrac{\begin{array}{c}\Gamma\\ \vdots\\ A\end{array}\qquad\begin{array}{c}\Delta\\ \vdots\\ A\end{array}}{A} \qquad\qquad r:\mathrm{Rep}\ \dfrac{\begin{array}{c}\Gamma\\ \vdots\\ A\end{array}}{A}$$

Rules of HNGL

Rules for NJ plus

$$r\!:^{k}\mathrm{Spt}_{\Gamma,\Delta}\ \dfrac{\begin{array}{c}{}^{k}[\Gamma],\Delta\\ \vdots\\ A\end{array}}{A}
\qquad\qquad
r\!:\mathrm{Com}_{A,B}\ \dfrac{\begin{array}{c}\Gamma\\ \vdots\\ A\end{array}}{B}$$

$$r\!:\mathrm{Ctr}\ \dfrac{\begin{array}{c}\Gamma\\ \vdots\\ A\end{array}\qquad\begin{array}{c}\Delta\\ \vdots\\ A\end{array}}{A}
\qquad\qquad
r\!:\mathrm{Rep}\ \dfrac{\begin{array}{c}\Gamma\\ \vdots\\ A\end{array}}{A}$$

A prederivation is a well-formed derivation tree based on the rules of HNGL.

Hyper rules

Applies to k prehyper deductions and produces another prehyper deduction:

$$\mathbf{h}\text{-}r \;\frac{R_1 \quad \cdots \quad R_k}{R}$$

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$$\mathbf{h}\text{-}r\ \dfrac{R_1 \quad \cdots \quad R_k}{R}$$

Hyper rule $\mathbf{h}$-r for NJ rule r

$$\mathbf{h}\text{-}{\to}\text{-e}\ \dfrac{R_1 \left| \begin{array}{c}\Gamma\\ \vdots\\ A \to B\end{array}\right. \qquad R_2 \left| \begin{array}{c}\Delta\\ \vdots\\ A\end{array}\right.}{R_1 \left| R_2 \left| \ {\to}\text{-e}\ \dfrac{\begin{array}{c}\Gamma\\ \vdots\\ A \to B\end{array} \quad \begin{array}{c}\Delta\\ \vdots\\ A\end{array}}{B}\right.\right.}$$

Hyper communication rule

$$\mathbf{h}\text{-Com}\ \dfrac{R_1 \;\Bigg|\; \begin{array}{c}\Gamma\\ \vdots\\ A\end{array} \qquad\qquad R_2 \;\Bigg|\; \begin{array}{c}\Delta\\ \vdots\\ B\end{array}}{R_1 \;\Bigg|\; R_2 \;\Bigg|\; \begin{array}{r}\Gamma\\ \vdots\\ x : \mathrm{Com}_{A,B}\,\dfrac{A}{B}\end{array} \;\Bigg|\; \begin{array}{r}\Delta\\ \vdots\\ \bar{x} : \mathrm{Com}_{B,A}\,\dfrac{B}{A}\end{array}}$$

Hyper splitting rule

$$\textbf{h}\text{-Spt}\ \dfrac{R \left|\begin{array}{c}\Gamma, \Delta\\ \vdots\\ A\end{array}\right.}{R \left|\begin{array}{c}{}^{k}[\Gamma], \Delta\\ \vdots\\ x{:}^{k}\,\mathrm{Spt}_{\Gamma,\Delta}\,\dfrac{A}{A}\end{array}\right|\begin{array}{c}\Gamma, {}^{l}[\Delta]\\ \vdots\\ \bar{x}{:}^{l}\,\mathrm{Spt}_{\Delta,\Gamma}\,\dfrac{A}{A}\end{array}}$$

Hyper contraction and repetition rules

$$\mathbf{h}\text{-Ctr}\ \genfrac{}{}{1pt}{0}{R \;\Bigg|\; \begin{array}{c}\Gamma\\ \vdots\\ A\end{array} \;\Bigg|\; \begin{array}{c}\Delta\\ \vdots\\ A\end{array}}{R \;\Bigg|\; x{:}\,\mathrm{Ctr}\,\dfrac{\begin{array}{c}\Gamma\\ \vdots\\ A\end{array}\quad\begin{array}{c}\Delta\\ \vdots\\ A\end{array}}{A}}
\qquad\qquad
\mathbf{h}\text{-Rep}\ \genfrac{}{}{1pt}{0}{R \;\Bigg|\; \begin{array}{c}\Gamma\\ \vdots\\ A\end{array}}{R \;\Bigg|\; x{:}\,\mathrm{Rep}\,\dfrac{\begin{array}{c}\Gamma\\ \vdots\\ A\end{array}}{A}}$$

Why this verbosity?

Natural deduction, as well as Sequent calculus, define a partial order of rule instances, and any linearisation that agrees with the partial order gives a valid derivation.

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Natural deduction, as well as Sequent calculus, define a partial order of rule instances, and any linearisation that agrees with the partial order gives a valid derivation.

In the case of Hyper Natural Deductions we have multiple trees with multiple partial orders, but due to the connections between prederivations via communication rules, the final HNGL does *not* define a unique derivation order.

Proof of linearity - GLC version

$C = (A \to B) \vee (B \to A)$,

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \Rightarrow A \qquad B \Rightarrow B}{A \Rightarrow B \mid B \Rightarrow A}\ com}{\Rightarrow A \to B \mid B \Rightarrow A}\ \to, r}{\Rightarrow A \to B \mid \ \Rightarrow B \to A}\ \to, r}{\Rightarrow C \mid \ \Rightarrow B \to A}\ \vee_1, r}{\Rightarrow C \mid \ \Rightarrow C}\ \vee_2, r}{\Rightarrow C}\ EC
$$

PROOF OF LINEARITY - HNGL VERSION

$$
\begin{prooftree}
\AxiomC{${}^1[A]$}
\LeftLabel{$x\colon \mathrm{Com}_{A,B}$}
\UnaryInfC{B}
\LeftLabel{${}^1\!\to\text{-}i$}
\UnaryInfC{$A \to B$}
\LeftLabel{$\vee\text{-}i$}
\UnaryInfC{C}
\AxiomC{${}^2[B]$}
\LeftLabel{$\overline{x\colon \mathrm{Com}_{A,B}}$}
\UnaryInfC{A}
\LeftLabel{${}^2\!\to\text{-}i$}
\UnaryInfC{$B \to A$}
\LeftLabel{$\vee\text{-}i$}
\UnaryInfC{C}
\LeftLabel{$y\colon \mathrm{Ctr}$}
\BinaryInfC{C}
\end{prooftree}
$$

HNGL DEDUCTION

$$A \qquad\qquad\qquad\qquad B$$

$\mathbf{h}$-Com

$$x{:}\,\mathrm{Com}_{A,B}\ \dfrac{A}{B} \;\Bigg|\; \bar{x}{:}\,\mathrm{Com}_{B,A}\ \dfrac{B}{A}$$

$\mathbf{h}$-$\to$-i

$$x{:}\,\mathrm{Com}_{A,B}\ \dfrac{{}^{1}[A]}{{}^{1}\ {\to}{:}\mathrm{i}\ \dfrac{B}{A \to B}} \;\Bigg|\; \bar{x}{:}\,\mathrm{Com}_{B,A}\ \dfrac{B}{A}$$

$\mathbf{h}$-$\to$-i

$$x{:}\,\mathrm{Com}_{A,B}\ \dfrac{{}^{1}[A]}{{}^{1}\ {\to}{:}\mathrm{i}\ \dfrac{B}{A \to B}} \;\Bigg|\; \bar{x}{:}\,\mathrm{Com}_{B,A}\ \dfrac{{}^{2}[B]}{{}^{2}\ {\to}{:}\mathrm{i}\ \dfrac{A}{B \to A}}$$

$\mathbf{h}$-$\vee$-i

$$x{:}\,\mathrm{Com}_{A,B}\ \dfrac{{}^{1}[A]}{{}^{1}\ {\to}{:}\mathrm{i}\ \dfrac{B}{\vee{:}\mathrm{i}\ \dfrac{A \to B}{C}}} \;\Bigg|\; \bar{x}{:}\,\mathrm{Com}_{B,A}\ \dfrac{{}^{2}[B]}{{}^{2}\ {\to}{:}\mathrm{i}\ \dfrac{A}{B \to A}}$$

$\mathbf{h}$-$\vee$-i

$$x{:}\,\mathrm{Com}_{A,B}\ \dfrac{{}^{1}[A]}{{}^{1}\ {\to}{:}\mathrm{i}\ \dfrac{B}{A \to B}} \;\Bigg|\; \bar{x}{:}\,\mathrm{Com}_{B,A}\ \dfrac{{}^{2}[B]}{{}^{2}\ {\to}{:}\mathrm{i}\ \dfrac{A}{B \to A}}$$

Results on HNGL

Theorem
If A is GLC *derivable, then A is also* HNGL *derivable.*

Theorem
If A is HNGL *derivable, then A is also* GLC *derivable.*

Theorem
The system HNGL *is sound and complete for infinitary propositional Gödel logic.*

Discussion

- Hyper rules – derivations are completely in ND style
- Hyper rules mimic HLK/BCF system
- natural style of deduction

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- Hyper rules – derivations are completely in ND style
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- but: procedural definition (like BCF system):
  - difficult to check whether a given figure forms a proof
  - difficult to reason on normalisation (needs reshuffling of proof trees)

We need criteria to check whether a set of trees forms a proof!

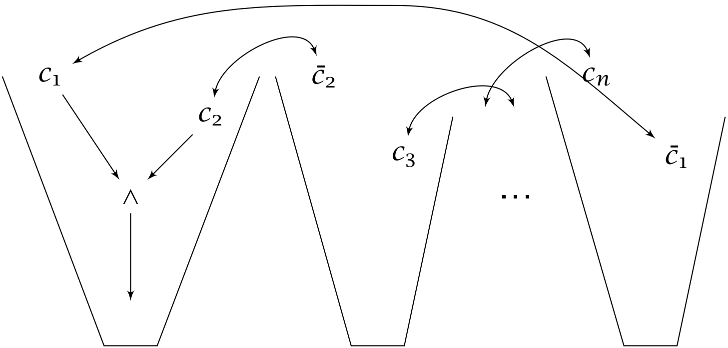
Towards an explicit definition

Proof criteria

What about the following proof part:

$$
\dfrac{
\dfrac{\vdots}{\dfrac{B}{A}\, x : \mathrm{Com}_{B,A}} \quad \vdots \quad E
\qquad
\dfrac{\vdots}{\dfrac{A}{B}\, \bar{x} : \mathrm{Com}_{A,B}} \quad \vdots \quad F
}{E \wedge F}
$$

Equivalence classes



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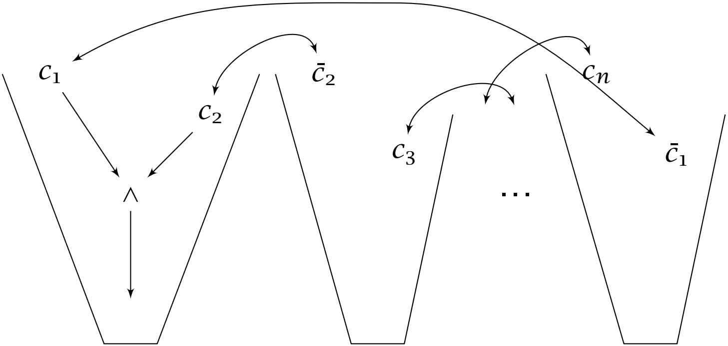

Criterion 1: The sets of trees connected to the sub-trees routed in the predecessors of any non-unary logical rule need to be disjoint.

Another criteria

What about this:

$$
\begin{array}{ccc}
\vdots & \qquad & \vdots \\
x : \mathrm{Com}_{B,A}\ \dfrac{B}{A} & \qquad & \bar{x} : \mathrm{Com}_{F,E}\ \dfrac{F}{E} \\
\vdots & \qquad & \vdots \\
x : \mathrm{Com}_{E,F}\ \dfrac{E}{F} & \qquad & \bar{x} : \mathrm{Com}_{A,B}\ \dfrac{A}{B} \\
\vdots & \qquad & \vdots
\end{array}
$$

Another criteria

What about this:

$$
\begin{array}{ccc}
\begin{array}{c}\vdots \\ x\colon \mathrm{Com}_{B,A}\ \dfrac{B}{A} \\ \vdots \\ x\colon \mathrm{Com}_{E,F}\ \dfrac{E}{F} \\ \vdots\end{array}
& \qquad &
\begin{array}{c}\vdots \\ \bar{x}\colon \mathrm{Com}_{F,E}\ \dfrac{F}{E} \\ \vdots \\ \bar{x}\colon \mathrm{Com}_{A,B}\ \dfrac{A}{B} \\ \vdots\end{array}
\end{array}
$$

Criterion 2: There is a total order on communication and split labels that is compatible with the order on the branches.

Canopy graphs

Two operations on labeled directed graphs:

$\mathrm{Cut}(\mathcal{G}, E)$	drops a set of edges from the graph
$\mathrm{Drop}(\mathcal{G}, N)$	drops a set of nodes and related edges that are reachable from all nodes labeled with a name in N

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Definition

Let $\mathcal{G} = (V, E, N, f)$ be a labeled graph, and let $E^c \subseteq E$ be the set of symmetric edges, that is the set of all edges $(r, s) \in E$ where also $(s, r) \in E$. If $\mathrm{Cut}(\mathcal{G}, E^c)$ is a disjoint union of trees, we call $\mathcal{G}$ a *C-graph* or *canopy graph*.

Motivation of these concepts

Consider the following hyper-sequent derivation:

$$
\dfrac{
  contr\;\dfrac{
    \wedge\text{-}r\;\dfrac{
      com_1\;\dfrac{\dfrac{B \Rightarrow B}{C, B \Rightarrow B} \qquad A \Rightarrow A}{C, B \Rightarrow A \mid A \Rightarrow B}
      \qquad
      com_2\;\dfrac{\dfrac{C \Rightarrow C}{C, B \Rightarrow C} \qquad A \Rightarrow A}{C, B \Rightarrow A \mid A \Rightarrow C}
    }{C, B \Rightarrow A \mid C, B \Rightarrow A \mid A \Rightarrow B \wedge C}
  }{C, B \Rightarrow A \mid A \Rightarrow B \wedge C}
}{\Rightarrow C \to (B \to A) \mid \; \Rightarrow A \to B \wedge C}
$$

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}{C, B \Rightarrow A \;\mid A \Rightarrow B \wedge C}
}{\Rightarrow C \to (B \to A) \mid\; \Rightarrow A \to B \wedge C}
$$

And the following intended HND proof:

$$
w\,\dfrac{z\,\dfrac{y{:}\mathrm{Ctr}\,\dfrac{x_1{:}\mathrm{Com}_{C,A}\,\dfrac{[C]}{A} \qquad x_2{:}\mathrm{Com}_{B,A}\,\dfrac{[B]}{A}}{A}}{B \to A}}{C \to (B \to A)}
$$

$$
v\,\dfrac{u{:}\wedge\text{-}i\,\dfrac{\bar{x}_1{:}\mathrm{Com}_{A,C}\,\dfrac{[A]}{C} \qquad \bar{x}_2{:}\mathrm{Com}_{A,B}\,\dfrac{[A]}{B}}{B \wedge C}}{A \to (B \wedge C)}
$$

Motivation of these concepts II

$$w\ \cfrac{z\ \cfrac{y\colon \mathrm{Ctr}\ \cfrac{x_1\colon \mathrm{Com}_{C,A}\ \cfrac{[C]}{A} \qquad x_2\colon \mathrm{Com}_{B,A}\ \cfrac{[B]}{A}}{A}}{B \to A}}{C \to (B \to A)}$$

$$v\ \cfrac{u\colon \wedge\text{-}i\ \cfrac{\bar{x}_1\colon \mathrm{Com}_{A,C}\ \cfrac{[A]}{C} \qquad \bar{x}_2\colon \mathrm{Com}_{A,B}\ \cfrac{[A]}{B}}{B \wedge C}}{A \to (B \wedge C)}$$

Motivation of these concepts II

$$
w\,\cfrac{z\,\cfrac{y{:}\mathrm{Ctr}\,\cfrac{x_1{:}\mathrm{Com}_{C,A}\,\cfrac{[C]}{A} \qquad x_2{:}\mathrm{Com}_{B,A}\,\cfrac{[B]}{A}}{A}}{B \to A}}{C \to (B \to A)}
$$

$$
v\,\cfrac{u{:}\wedge\text{-}i\,\cfrac{\bar{x}_1{:}\mathrm{Com}_{A,C}\,\cfrac{[A]}{C} \qquad \bar{x}_2{:}\mathrm{Com}_{A,B}\,\cfrac{[A]}{B}}{B \wedge C}}{A \to (B \wedge C)}
$$

and the associated graph

$$
\begin{tikzcd}
x_1 \arrow[dr] \arrow[rrrrrr, leftrightarrow, bend left] & & x_2 \arrow[dl] \arrow[rr, leftrightarrow] & & \bar{x}_2 \arrow[dr] & & \bar{x}_1 \arrow[dl] \\
& y \arrow[r] & z \arrow[r] & w & & u \arrow[r] & v
\end{tikzcd}
$$

Motivation of these concepts II

$$
w\,\dfrac{z\,\dfrac{y{:}\mathrm{Ctr}\,\dfrac{x_1{:}\mathrm{Com}_{C,A}\,\dfrac{[C]}{A} \qquad x_2{:}\mathrm{Com}_{B,A}\,\dfrac{[B]}{A}}{A}}{B \to A}}{C \to (B \to A)}
$$

$$
v\,\dfrac{u{:}\wedge\text{-}i\,\dfrac{\bar{x}_1{:}\mathrm{Com}_{A,C}\,\dfrac{[A]}{C} \qquad \bar{x}_2{:}\mathrm{Com}_{A,B}\,\dfrac{[A]}{B}}{B \wedge C}}{A \to (B \wedge C)}
$$

connectivity condition does not hold for u

$$
\begin{tikzcd}
x_1 \arrow[rrr, leftrightarrow, bend left] \arrow[dr] & & x_2 \arrow[dl] \arrow[r, leftrightarrow] & \bar{x}_2 & \bar{x}_1 \\
& y \arrow[r] & z \arrow[r] & w &
\end{tikzcd}
$$

Motivation of these concepts II

$$
\dfrac{
  \dfrac{
    x_1{:}\,\mathrm{Com}_{C,A}\ \dfrac{[C]}{A}
    \qquad
    x_2{:}\,\mathrm{Com}_{B,A}\ \dfrac{[B]}{A}
  }{
    \dfrac{A}{B \to A}\ z
  }\ y{:}\,\mathrm{Ctr}
}{
  C \to (B \to A)
}\ w
$$

$$
\dfrac{
  \dfrac{
    \bar{x}_1{:}\,\mathrm{Com}_{A,C}\ \dfrac{[A]}{C}
    \qquad
    \bar{x}_2{:}\,\mathrm{Com}_{A,B}\ \dfrac{[A]}{B}
  }{
    B \wedge C
  }\ u{:}\wedge\text{-}i
}{
  A \to (B \wedge C)
}\ v
$$

cut at the contraction, conn. comp. fall apart

$$
\begin{tikzcd}
x_1 \arrow[rrr, bend left, leftrightarrow] & x_2 \arrow[r, leftrightarrow] \arrow[d] & \bar{x}_2 & \bar{x}_1 \\
& y \arrow[r] & z \arrow[r] & w
\end{tikzcd}
$$

Motivation of these concepts II

$$
w\,\dfrac{z\,\dfrac{y{:}\mathrm{Ctr}\,\dfrac{x_1{:}\mathrm{Com}_{C,A}\,\dfrac{[C]}{A} \qquad x_2{:}\mathrm{Com}_{B,A}\,\dfrac{[B]}{A}}{A}}{B \to A}}{C \to (B \to A)}
$$

$$
v\,\dfrac{u{:}\wedge\text{-}i\,\dfrac{\bar{x}_1{:}\mathrm{Com}_{A,C}\,\dfrac{[A]}{C} \qquad \bar{x}_2{:}\mathrm{Com}_{A,B}\,\dfrac{[A]}{B}}{B \wedge C}}{A \to (B \wedge C)}
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$$
\begin{tikzcd}
x_1 \arrow[rrrr, leftrightarrow, bend left] & x_2 \arrow[rr, leftrightarrow] \arrow[d] & & \bar{x}_2 & \bar{x}_1 \\
y \arrow[r] & z \arrow[r] & w & &
\end{tikzcd}
$$

Expresses an implicit ordering between the conjunction (introduced first) and the contraction (introduced later).

Explicit definition of HND for Gödel logics

A finite set of pre-derivations R (together with a total order on labels) forms a hyper natural deduction iff

- some obvious consistency conditions are satisfied;
  like occurrence of dual labels, compatibility with fixed label order, ...

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Explicit definition of HND for Gödel logics

A finite set of pre-derivations R (together with a total order on labels) forms a hyper natural deduction iff

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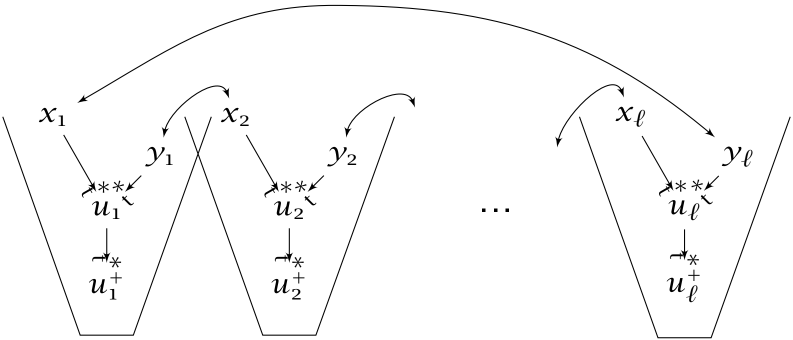
- Independence of premises for non-unary logical rules r and communication: The connected components in $\mathrm{Cut}(\mathrm{Drop}(\mathcal{G}(R), r))$ of premises of r are disjoint.
- Local dependence of contraction premises r: The connected components in $\mathrm{Cut}(\mathrm{Drop}(\mathcal{G}(R), r))$ of premises of r are equal.

Core lemma

Chain lemma – in a GLHD the following figure cannot appear.



Normalisation

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$$
{\to}\text{-e}\ \dfrac{{\to}\text{-i}\ \dfrac{\begin{array}{c}[A]\\ \vdots\\ B\end{array}}{A \to B} \qquad \begin{array}{c}\Gamma\\ \vdots\\ A\end{array}}{B}
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Normalisation

Idea: Reorder deductions where an introduction rule is followed by an elimination rule:

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\qquad \text{converts to} \qquad
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\qquad \text{converts to} \qquad
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$$

Effect of normalisation: hourglass form of derivation, eliminations followed by introductions.

Permutation Conversions for hyper natural deduction

Example conversion for normalisation in hyper natural deduction:

$$
x{:}\mathrm{Com}_{A\to B,C}\ \dfrac{\begin{array}{c}\Gamma\\ \sigma_0\,\vdots\\ A\to B\end{array}}{C}
\qquad
\bar{x}{:}\mathrm{Com}_{C,A\to B}\ \dfrac{\dfrac{\begin{array}{c}\Delta\\ \sigma_1\,\vdots\\ C\end{array}}{A\to B}\quad \begin{array}{c}\Pi\\ \sigma_2\,\vdots\\ A\end{array}}{B}\,{\to}\text{-}e
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first try: use dual labels as channels to communicate sub-derivations

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$$x{:}\mathrm{Com}_{A\to B,C}\ \frac{\begin{array}{c}{\color{red}\Gamma}\\ \sigma_0\ \vdots\\ A\to B\end{array}}{C} \qquad \bar{x}{:}\mathrm{Com}_{C,A\to B}\ \begin{array}{c} \\ \\ \to\text{-}e\end{array}\frac{\dfrac{\begin{array}{c}\Delta\\ \sigma_1\ \vdots\\ C\end{array}}{A\to B} \qquad \begin{array}{c}\Pi\\ \sigma_2\ \vdots\\ A\end{array}}{B}$$

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converts to (similar to cut-elimination in HLK)

$$x{:}\mathrm{Com}_{B,C}\ \dfrac{{}^{1}S^{y}_{\Gamma,\Pi}\ \dfrac{\to\text{-}e\ \dfrac{\begin{array}{c}\Gamma\\ \sigma_0\,\vdots\\ A\to B\end{array}\quad \begin{array}{c}{}^{1}[\Pi]\\ \sigma_2\,\vdots\\ A\end{array}}{B}}{B}}{C}
\qquad
\mathit{contr}\ \dfrac{{}^{2}S^{\bar{y}}_{\Pi,\Gamma}\ \dfrac{\to\text{-}e\ \dfrac{\begin{array}{c}{}^{2}[\Gamma]\\ \sigma_0\,\vdots\\ A\to B\end{array}\quad \begin{array}{c}\Pi\\ \sigma_2\,\vdots\\ A\end{array}}{B}}{B}\qquad \bar{x}{:}\mathrm{Com}_{C,B}\ \dfrac{\begin{array}{c}\Delta\\ \sigma_1\,\vdots\\ C\end{array}}{B}}{B}$$

Conversions

- proof follows Troelstra/Schwichtenberg proof
- detour conversions, simplification conversion and permutation conversions as there, with cases for cut and split added
- branches and tracks
- double induction on cut-rank and ordinal sum of critical label sequences

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Theorem

Contraction, communication and splitting permutation conversions convert hyper natural deductions into hyper natural deductions.

Results

Theorem (Normalisation)

Hyper Natural Deduction for Gödel Logic admits (weak) normalisation. That is, there is a way to move all elimination rules above introduction rules by applying the above conversions.

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Theorem (Sub-formula property)

Let R be a normal hyper natural deduction with derived hypersequent $\mathcal{H}$. Then each formula in R is a subformula of a formula in $\mathcal{H}$.

Returning to our wishlist

(semi) local

- ✔ construction of deductions:
  apply ND inspired rules to extend a HND deductions

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normalisation

- ✔ procedural normalisation via conversion steps

Further steps

- Extend hyper natural deduction to first order
- Reconsidering BCF system in the light of our procedural definition
- Develop term systems (“parallel λ”) and establish Curry-Howard correspondences
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Thanks for your attention!

Ref: Beckmann, A. and P., N. *Hyper Natural Deductions*, to appear in *Journal of Logic and Computation.*