On the mathematical and foundational significance of the uncountable

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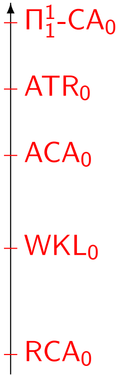
This talk: introducing higher-order objects destroys the 'Big Five' picture of RM and collapses the associated linear order.

The Big Five picture of RM



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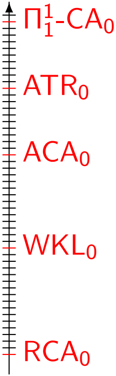

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ATR_0

ACA_0

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RCA_0 proves Interm. value thm, Soundness thm, Existence of alg. clos. ...

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$WKL_0 \leftrightarrow$ Peano exist. $\leftrightarrow$ Weierstraß approx. $\leftrightarrow$ Weierstraß max. $\leftrightarrow$ Hahn-Banach $\leftrightarrow$ Heine-Borel $\leftrightarrow$ Brouwer fixp. $\leftrightarrow$ Gödel compl. $\leftrightarrow \dots$

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This talk: introducing higher-order objects destroys the 'Big Five' picture and collapses the linear order; the picture and order are merely artefacts of second-order arithmetic (in particular: of countable approximations).

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The Cousin and Lindelöf lemmas cannot be formalised in second-order arithmetic.

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YES; even in scientifically applicable math!

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The development of the gauge integral:
Denjoy-Luzin-Perron-Henstock-Kurzweil

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Z_2^ω

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All the second-order systems have higher-order counterparts!

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Z_2^ω $(\exists^3)$: there is a functional $\exists^3$ deciding '$(\exists f \in \mathbb{N}^\mathbb{N})(F(f) = 0)$' for any F^2

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WKL_0^ω (FF): the fan functional computes a modulus of uniform continuity for any continuous functional on $2^\mathbb{N}$

$\mathsf{RCA}_0^\omega/\mathsf{RCA}_0$

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PS: Borel's proof of HBU ($\approx$ 1900) makes no use of the axiom of choice. With minimal adaption, Borel's proof yields a realiser Θ for HBU.

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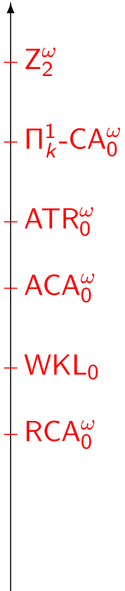

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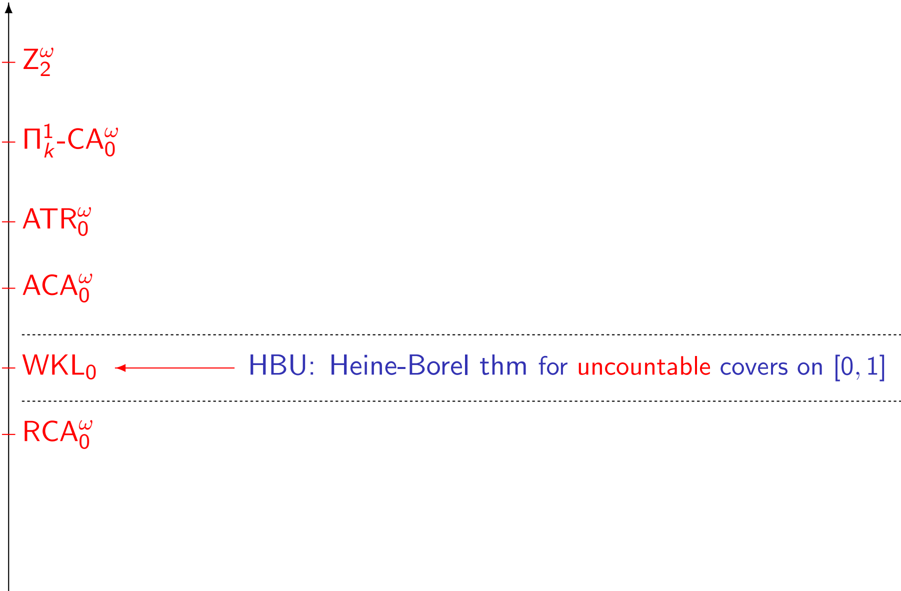

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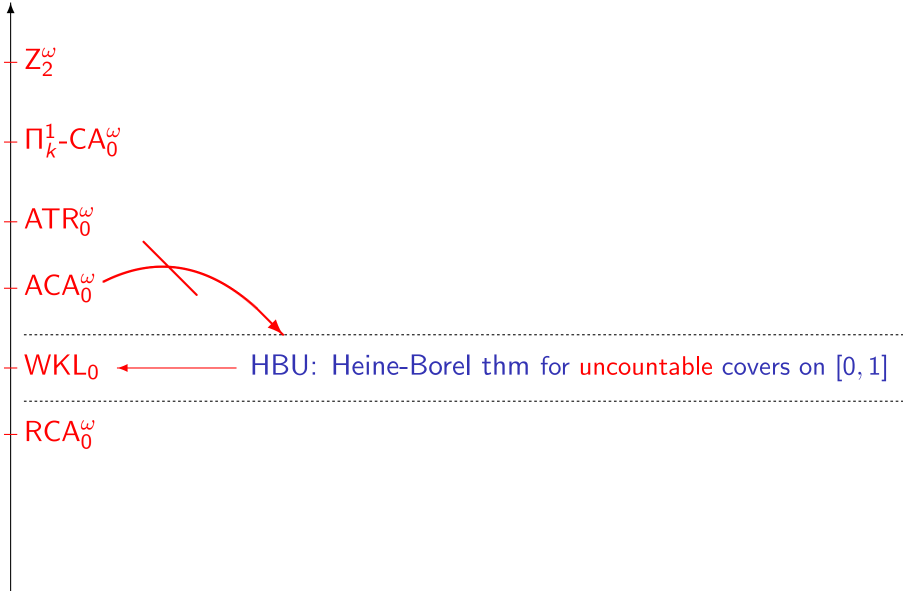

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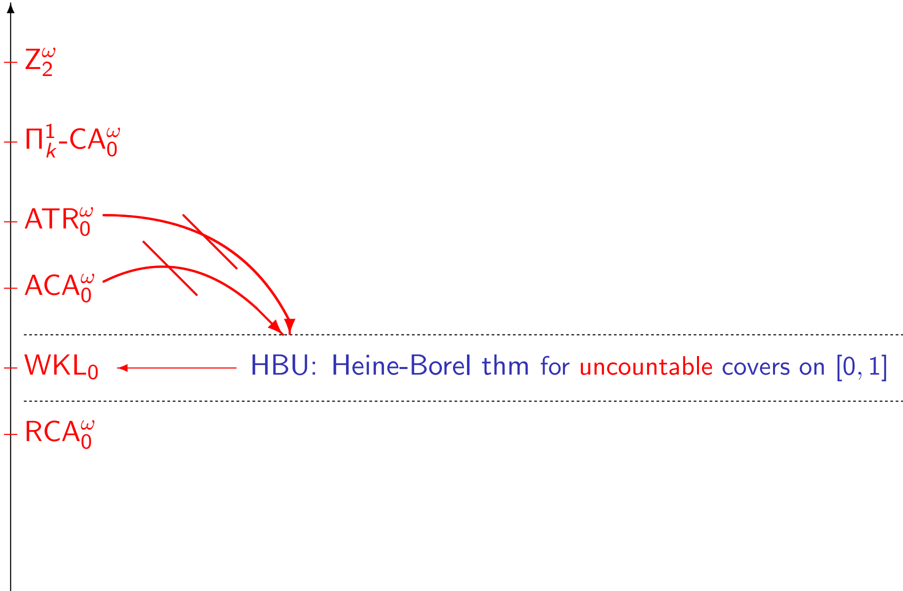

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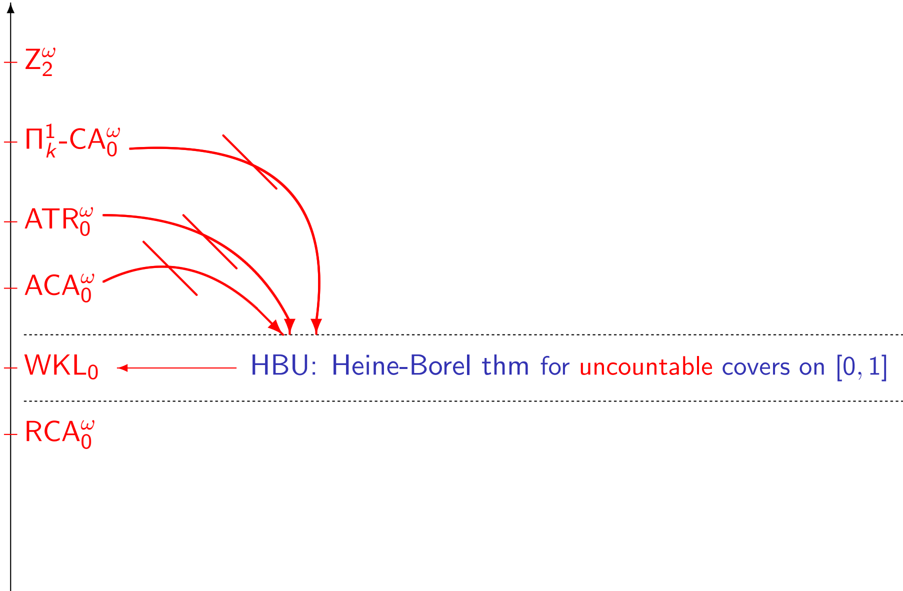

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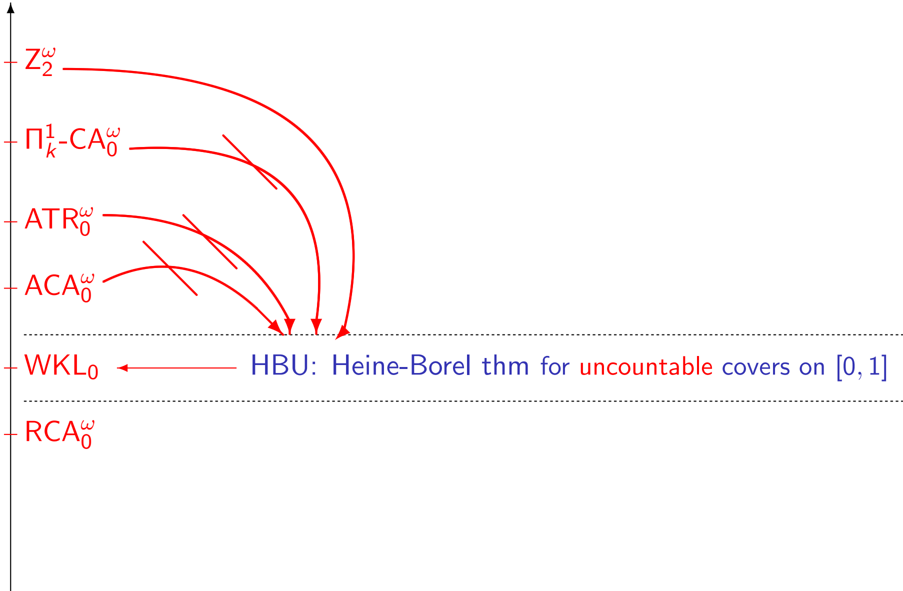

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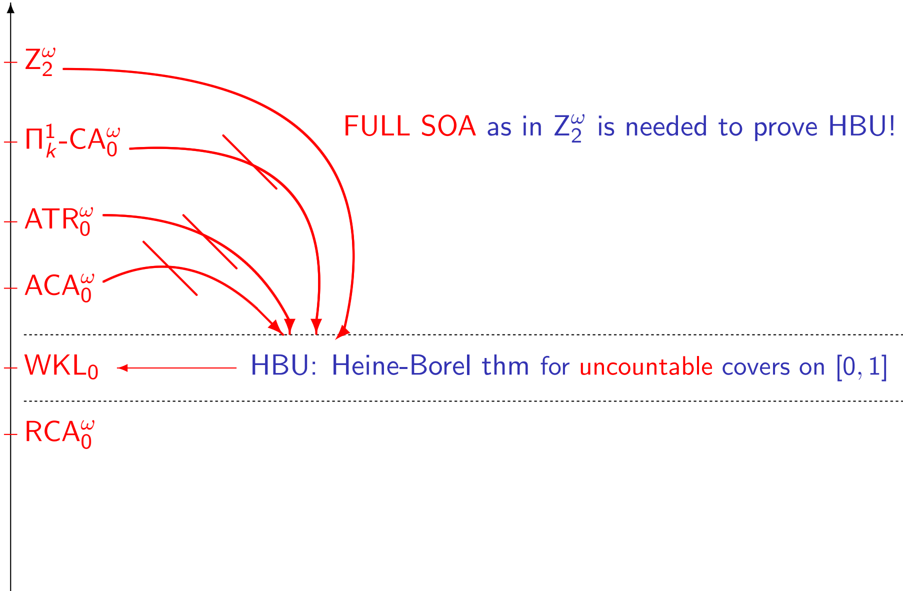

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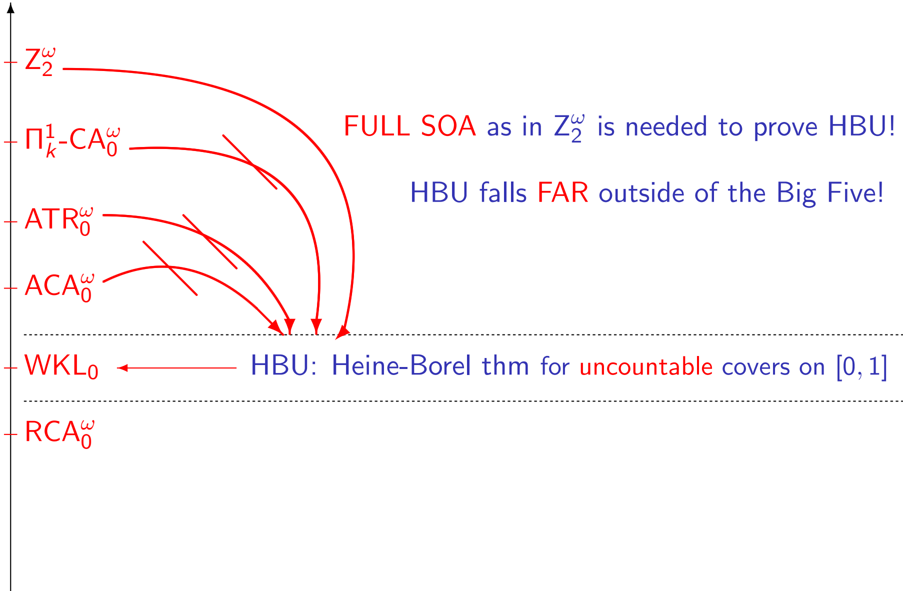

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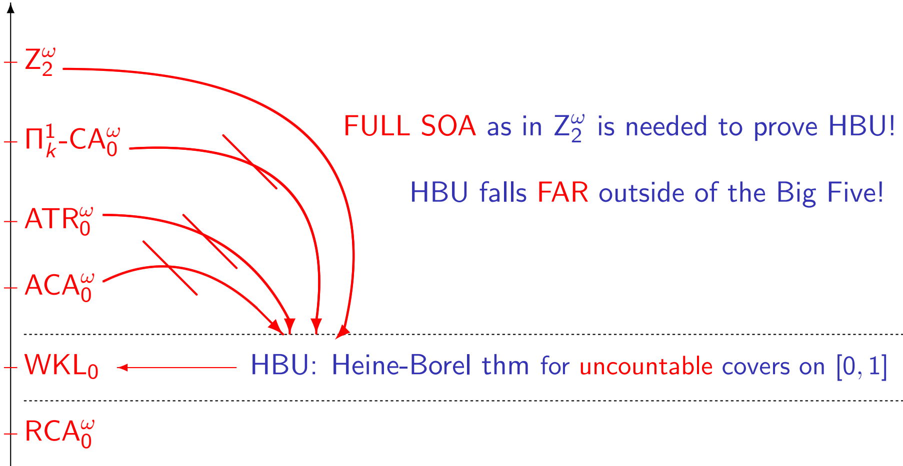

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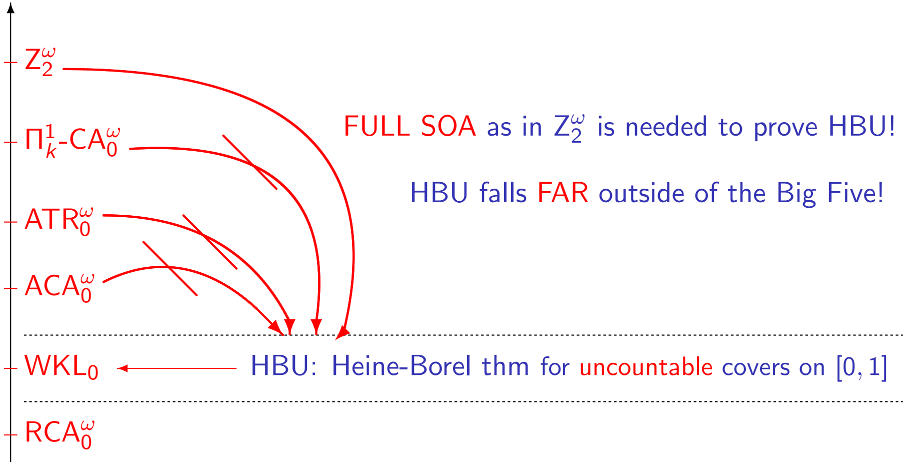

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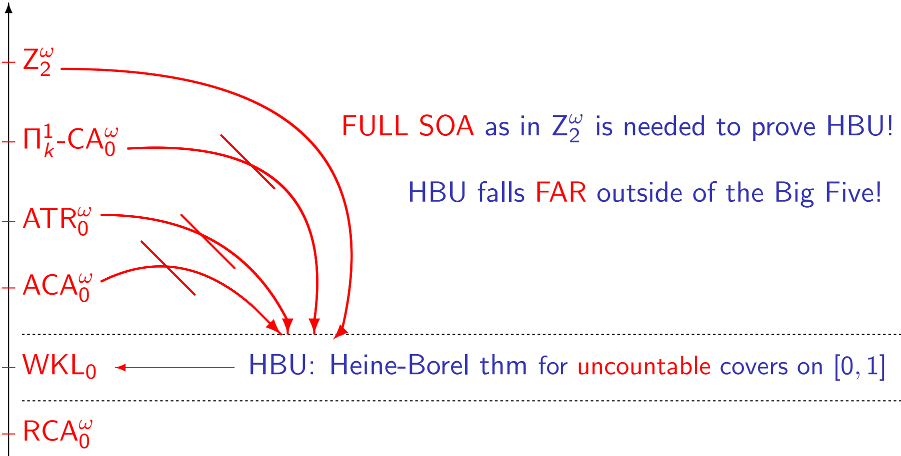

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$P = (0, t_1, x_1, \ldots x_k, t_k, 1)$ partition of I; mesh $\|P\| := \max_{i \leq k}(x_{i+1} - x_i)$; Riemann sum $S(P,f) = \sum_{i=0}^{k} f(t_i)(x_{i+1} - x_i)$.

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Theorems of second-order arithmetic NEVER jump anywhere!

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Anil Nerode: That's not reverse math, that's topsy turvy math!

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And many more: the dam really breaks!

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The original Bolzano-Weierstrass thm has produced many such 'uniform' theorems of considerable hardness (namely requring Z_2^ω). Weierstrass' version of the Bolzano-Weierstrass thm was 'more constructive' (requiring only ACA_0)'; the former was forgotten by history....

Paper

Most of the aforementioned results are proved in:

On the mathematical and foundational significance of the uncountable (*Dag Normann & Sam Sanders, arXiv*)

https://arxiv.org/abs/1711.08939

This paper makes NO use of Nonstandard Analysis.

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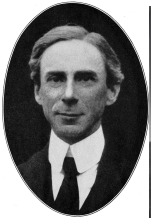  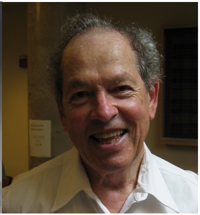

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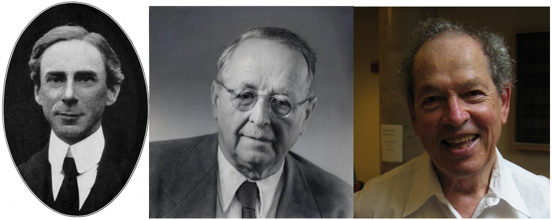

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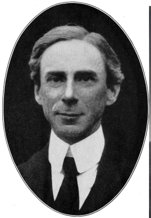

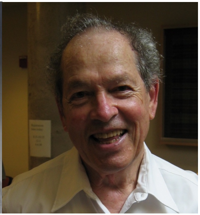

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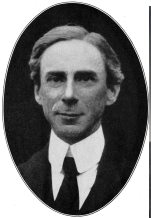

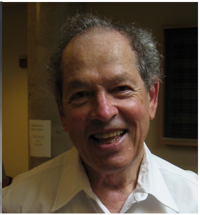

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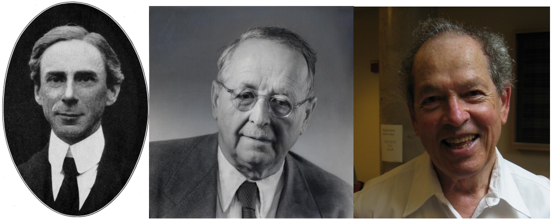

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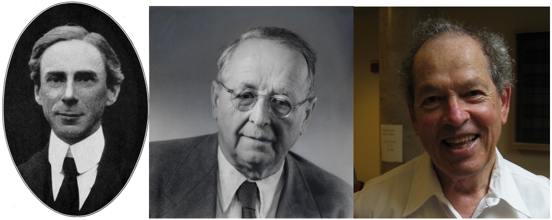

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Anil Nerode: Bishop said we should not try to formalise his notion of 'constructive'; these results suggest that Bishop was right!

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Most results have two proofs: one via NSA and S_{st} (weak base theory; terms of Gödel's T), and one via higher-order recursion theory (more general results, greater scope). Most of the above is both Normann-S.

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Thank you for your attention!